Friday, 10 March 2023 11:56

Ex 4.7.5

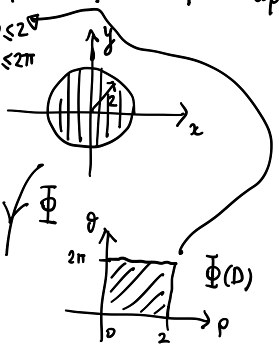
#1 Compute $\underline{\int_{\underline{x^2+y^2\le 4}} \sqrt{4-x^2-y^2}\,dxdy}$

Sol: $\int_{\underline{x^2+y^2\le 4}} \sqrt{4-(x^2+y^2)}\,dx\,dy$

$= \int \sqrt{4-\rho^2}\cdot\rho\,d\rho\,d\theta$

$\phi^{-1}\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $0\le\rho\le 2$, $0\le\theta\le 2\pi$

$x^2+y^2=\rho^2$



$\overset{RF}{=} \int_0^2 \int_0^{2\pi} \underbrace{\rho(4-\rho^2)^{1/2}}\cdot 1\,d\theta\;d\rho$

$= 2\pi\int_0^2 \rho(4-\rho^2)^{1/2}\,d\rho$

$\partial_\rho \frac{(4-\rho^2)^{3/2}}{-3} = \frac{\cancel{\boxed{-3}}}{\cancel{2}}(4-\rho^2)^{1/2}\cdot(+\cancel{2}\rho)$

$= \frac{2\pi}{-3}\left[\frac{(4-\rho^2)^{3/2}}{\cancel{\boxed{-3}}}\right]_{\rho=0}^{\rho=2}$

$$= -\frac{2\pi}{3}\left[\,0 - 4^{3/2}\right] = \frac{2\pi}{3}\,8 = \frac{16\pi}{3}$$ ■

#2 $$\int_{\boxed{x^2+y^2\le 1}} \frac{1}{1+x^2+y^2}\,dx\,dy$$

$$= \int_{\substack{0\le\rho\le 1\\ 0\le\theta\le 2\pi}} \frac{1}{1+\rho^2}\,\rho\,d\rho\,d\theta$$

$$\phi^{-1}\begin{cases} x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$$

$$\rho^2\le 1 \qquad \{(\rho,\theta)\in[0,+\infty[\times[0,2\pi] : \rho^2\le 1\}$$

$$\Updownarrow$$

$$0\cancel{-1}\le\rho\le 1$$

$$\overset{RF}{=} \int_0^1 \int_0^{2\pi} \boxed{\frac{\rho}{1+\rho^2}}\cdot 1\,d\theta\,d\rho$$

$$= \int_0^1 \frac{\rho}{1+\rho^2} \underbrace{\int_0^{2\pi} 1\,d\theta}_{2\pi}\,d\rho$$

$$= \cancel{2}\pi \int_0^1 \underbrace{\frac{2\rho}{1+\rho^2}}_{\partial_\rho \log(1+\rho^2)}\,d\rho = \pi\left[\log(1+\rho^2)\right]_{\rho=0}^{\rho=1}$$

$$= \pi\log 2.$$ ■

#3 $$\int_{\mathbb{R}^2} e^{-\overset{x^2+y^2+y^2}{(x^2+2y^2)}}\,dx\,dy =$$

$$\underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \int_{\substack{\rho\geq 0\\ 0\leq\theta\leq 2\pi}} e^{-(\rho^2+\rho^2\underline{\sin^2\theta})}\,\rho\, d\rho d\theta$$

$$= \int_{\rho\geq 0,\ 0\leq\theta\leq 2\pi} e^{-(1+\sin^2\theta)\rho^2}\,\rho\, d\rho d\theta$$

$$\overset{RF}{=} \int_0^{2\pi}\left(\int_0^{+\infty} \underbrace{e^{-(1+\sin^2\theta)\rho^2}\,\rho}\, d\rho\right) d\theta$$

$$\partial_\rho\, \frac{e^{-(1+\sin^2\theta)\rho^2}}{-2(1+\sin^2\theta)} = e^{-(1+\sin^2\theta)\rho^2}\underbrace{\cancel{\left(-2(1+\sin^2\theta)\right)}}\rho$$

$$= \int_0^{2\pi}\left[\frac{e^{-(1+\sin^2\theta)\rho^2}}{\underbrace{-2(1+\sin^2\theta)}}\right]_{\rho=0}^{\rho=+\infty} d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2(1+\sin^2\theta)}\cdot\left(0-1\right)\, d\theta$$

$$= +\frac{1}{2}\int_0^{2\pi}\frac{1}{1+\sin^2\theta}\, d\theta \qquad \text{too long}$$

$$\int_{\mathbb{R}^2} e^{-(x^2+2y^2)}\, dxdy =$$

$$x^2+2y^2=\rho^2$$

$$x^2+(\sqrt{2}\,y)^2=\rho^2$$

$$x = \rho\cos\theta$$
$$\sqrt{2}y = \rho\sin\theta \quad \Leftrightarrow \quad \phi^{-1}\boxed{\begin{cases} x = 1\rho\cos\theta \\ y = \dfrac{1\rho}{\sqrt{2}\cancel{2}}\sin\theta \end{cases}}$$

$$= \int_{\substack{\rho\geq 0 \\ 0\leq\theta\leq 2\pi}} e^{-\rho^2} \cdot \underbrace{|\det(\phi^{-1})'|}_{\frac{1}{\sqrt{2}}\cdot\rho} \, d\rho\, d\theta$$

$$\phi^{-1}(\rho,\theta) = \left(\rho\cos\theta \quad , \frac{\rho}{\sqrt{2}}\sin\theta\right)$$

In general, we can consider

$$\phi^{-1} : \begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases} \qquad a, b > 0$$

$$\phi^{-1}(\rho,\theta) = (\overbrace{a\ \rho\cos\theta}^{\phi_1^{-1}},\ \overset{\phi_2^{-1}}{b\rho\sin\theta})$$

$$\det(\phi^{-1})'(\rho,\theta) \overset{=}{\det}\begin{bmatrix} \nabla\phi_1^{-1} \\ \nabla\phi_2^{-1} \end{bmatrix}$$

$$= ab\det\begin{bmatrix} \text{jac. matrix of pol coord} \end{bmatrix}$$

$$= ab\rho.$$

$$|\det(\phi^{-1})'| = ab\rho.$$

$$= \int_{\rho\geq 0} e^{-\rho^2}\frac{\rho}{\sqrt{2}}\, d\rho\, d\theta$$

$0 \leq \theta \leq 2\pi$

$$\overset{RF}{=} -\frac{1}{\sqrt{2}} \cancel{2}\pi \int_0^{+\infty} \underbrace{-2\rho e^{-\rho^2}}_{\partial_\rho e^{-\rho^2}} d\rho$$

$$= -\frac{\pi}{\sqrt{2}} \underset{0 \quad -1}{\left[e^{-\rho^2} \right]_0^{+\infty}} = \frac{\pi}{\sqrt{2}}. \qquad \blacksquare$$

Ex 4.7.3

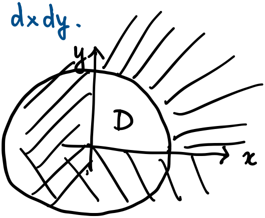
$$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq r^2\}$$

Q1 Draw D

Q2 Compute barycentre of D

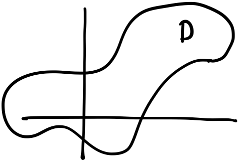
Q3 $\displaystyle\int_D \frac{x+y}{x^2+y^2} \, dx\, dy.$

Sol: Q1



Q2.

Def: Given $D \subset \mathbb{R}^2$, the barycentre of D is a point $(\bar{x}, \bar{y})$



where $\bar{x}$ is the average x of pts of D
and $\bar{y}$ " " " y " " "

$$\bar{x} := \frac{1}{\text{Area}\, D}\int_D x \; dx\,dy$$

$$\bar{y} := \frac{1}{\text{Area}\, D}\int_D y \; dx\,dy$$

Q2 . Area D $= \frac{1}{4}\pi r^2$

$$\int_D x \; dx\,dy = \int_{\boxed{x\geq 0,\, y\geq 0,\, x^2+y^2\leq r^2}} x \; dx\,dy$$

$$\overset{\text{RF}}{=} \int_0^r \int_0^{\sqrt{r^2-x^2}} x\cdot 1 \; dy \; dx$$

$$\boxed{0 \leq y} \leq \sqrt{r^2-x^2}$$

$$\boxed{y^2 \leq \underbrace{r^2-x^2}_{}} \iff -\sqrt{r^2-x^2} \leq y \leq \sqrt{r^2-x^2}$$

$$\Longrightarrow r^2-x^2 \geq 0 \iff x^2 \leq r^2$$

$$0 \leq x \leq r \qquad \iff -r \leq x \leq r$$

$$= \int_0^r x\sqrt{r^2-x^2} \; dx$$

$$= \int_0^r (\underbrace{r^2-x^2}_{})^{1/2} \underbrace{x}_{} \; dx$$

$$\partial_x \frac{(r^2-x^2)^{3/2}}{-3} = \cancel{-\frac{3}{2}}(r^2-x^2)^{1/2}(+\cancel{2}x)$$

$$= \left[\frac{(r^2-x^2)^{3/2}}{\underbrace{-3}}\right]_{x=0}^{x=r}$$

$$= -\frac{1}{3}\left(0 - (r^2)^{3/2}\right) = \frac{r^3}{3} . \checkmark$$

$$\bar{x} = \frac{1}{\frac{\pi r^2}{4}} \cdot \frac{r^3}{3} = \frac{4}{3\pi} r.$$

Alt sol : $\displaystyle\int_D x \, dx \, dy \underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=}$

$$\int_{\substack{0\le\rho\le r \\ 0\le\theta\le\pi/2}} \rho^2\cos\theta \cdot \cancel{\rho} \; d\rho \, d\theta$$

$$\overset{RF}{=} \int_0^r \rho^2 \underbrace{\int_0^{\pi/2} \cos\theta \, d\theta}_{\parallel} \; d\rho$$

$$[\sin\theta]_0^{\pi/2} = 1 - 0 = 1$$

$$= \int_0^r \rho^2 \, d\rho = \frac{\rho^3}{3}\Big|_0^r = \frac{r^3}{3}$$

About y : $\displaystyle\int_{x\geq 0,\, y\geq 0,\, x^2+y^2\leq r^2} y \, dx\,dy$

$$= \int_{x\geq 0,\, y\geq 0 \;\; x^2+y^2\leq r} x \, dx dy \qquad = \frac{r^3}{3}$$

$$\Rightarrow \bar{y} = \bar{x}.$$

Q3. $\displaystyle\int_D \frac{x+y}{x^2+y^2} \, dx\,dy \underset{\text{pol coords}}{=}$

$$= \int_{\substack{0\leq \rho \leq r \\ 0 \leq \theta \leq \pi/2}} \frac{\cancel{\rho}\cos\theta + \cancel{\rho}\sin\theta}{\cancel{\rho^2}} \cancel{\rho}^2 d\rho d\theta$$

$$\overset{RF}{=} \int_0^r \underbrace{\left(\int_0^{\pi/2} \cos\theta + \sin\theta \, d\theta \right)} d\rho$$

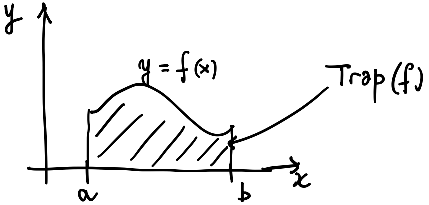
$$= r \cdot \left(\underset{1-0}{[\sin\theta]_0^{\pi/2}} + \underset{-(0-1)}{[+\cos\theta]_0^{\pi/2}} \right) \qquad = 2$$

$$= 2r. \quad \blacksquare$$

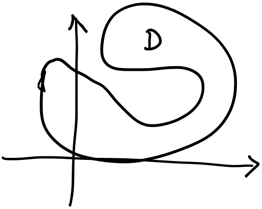
General Area and Measure of a set

$f : [a,b] \longrightarrow [0, +\infty[$

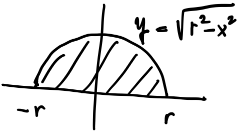
$$\int_a^b f(x)\,dx = \text{Area } \text{Trap}(f)$$

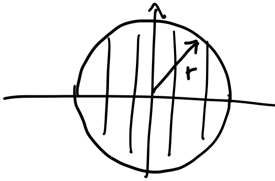


In this way, we can compute the area of many plane figures, however we are limited to areas of type Trap (f)



Area D ?





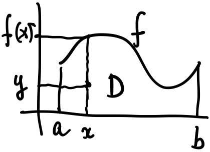
$$\begin{array}{c} 2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx \\ \| \\ \text{Area Disk} \end{array}$$

Def:

$$\text{Area } D := \int_D 1 \, dx\, dy \qquad D \subset \mathbb{R}^2$$

Rmk: If $D = \text{Trap}\, f$

$f : [a,b] \longrightarrow [0,+\infty[$



$$\text{Area } D = \int_D 1 \, dx\, dy \qquad D = \left\{ (x,y) : \begin{array}{l} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{array} \right\}$$

$$= \int_{\substack{a \leq x \leq b \\ 0 \leq y \leq f(x)}} 1 \quad dx\, dy$$

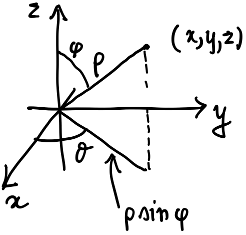
$$\overset{RF}{=} \int_a^b \underbrace{\int_0^{f(x)} 1 \quad dy}_{\| \; f(x)} \quad dx$$

$$\int_\alpha^\beta 1 \, dy = \beta - \alpha$$

$$= \int_a^b f(x)\, dx \qquad \blacksquare$$

$$\text{Def:} \qquad \text{Vol } D := \int_D 1 \quad dx\,dy\,dz \qquad D \subset \mathbb{R}^3$$

Spherical Coords



$$\Phi^{-1} \left\{ \begin{array}{l} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{array} \right.$$

$$\rho \geq 0, \; 0 \leq \varphi \leq \pi, \; 0 \leq \theta \leq 2\pi$$

$$\int f(x,y,z)\; dx\,dy\,dz =$$

D

$$= \int_{\Phi(D)} f\left(\overset{x}{\rho \sin\varphi\cos\theta}, \overset{y}{\rho\sin\varphi\sin\theta}, \rho\cos\varphi\right) \cdot$$

$$\cdot \left|\det(\phi^{-1})'\right| d\rho\, d\varphi\, d\theta$$

$\det(\phi^{-1})'$

$s = \sin\varphi$ $\quad S = \sin\theta$
$c = \cos\varphi$ $\quad C = \cos\theta$

$$\phi^{-1}(\rho,\varphi,\theta) = \left(\boxed{\rho\sin\varphi\cos\theta}, \boxed{\rho\sin\varphi\sin\theta}, \boxed{\rho\cos\varphi}\right)$$

$$\det(\phi^{-1})' = \det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\varphi & \partial_\theta \end{array} \\ \begin{bmatrix} sC & \rho c C & -\rho s S \\ sS & \rho c S & \rho s C \\ c & -\rho s & 0 \end{bmatrix} \end{array}$$

$$= \rho^2 c^2 s \det \begin{bmatrix} \cancel{\rho}\cancel{c}C & -\cancel{\rho}\cancel{s}S \\ \cancel{\rho}\cancel{c}S & \cancel{\rho}\cancel{s}C \end{bmatrix} = C^2 + S^2 \equiv 1$$

$$+ (+\rho^2 s^3) \det \begin{bmatrix} \cancel{s}C & -\cancel{\rho}\cancel{s}S \\ \cancel{s}S & \cancel{\rho}\cancel{s}C \end{bmatrix} = 1$$

$$= \rho^2 c^2 s + \rho^2 s^3 = \rho^2 s \underset{\overset{\shortparallel}{1}}{(c^2 + s^2)}$$

$$= \rho^2 s$$

$$\boxed{|\det(\phi^{-1})'| = \rho^2 \underset{\geq 0}{\sin\varphi}} \qquad \varphi \in [0,\pi]$$

Ex. 4.4.5 Volume of sphere of radius r

Sol: $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq r^2\}$

$$\text{Vol } D = \int_D 1 \, dx\, dy\, dz$$

$$= \int_{\underline{x^2+y^2+z^2} \leq r^2} 1 \quad dx\, dy\, dz$$

$$\underset{\text{sph. coords}}{=} \int 1 \cdot \rho^2 \sin\varphi \quad d\rho\, d\varphi\, d\theta$$

$$\cancel{-r \leq} \rho \leq r \Longleftarrow \rho^2 \leq r^2 \iff 0 \leq \rho \leq r$$

$$0 \leq \varphi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

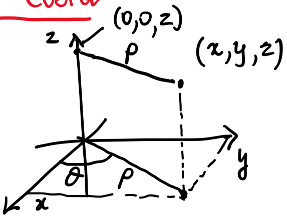
$$= \int_0^r \left(\int_0^{2\pi} \left(\int_0^{\pi} \rho^2 \overset{\overset{\partial_\varphi(-\cos\varphi)}{\parallel}}{\sin\varphi} \, d\varphi \right) d\theta \right) d\rho$$

$$= \int_0^r \rho^2 \int_0^{2\pi} \underbrace{[-\cos\varphi]_{\varphi=0}^{\varphi=\pi}}_{\underbrace{+1+1}_{2}} d\theta \, d\rho$$

$$= 2 \int_0^r \rho^2 \underbrace{\int_0^{2\pi} 1 \, d\theta}_{2\pi} \, d\rho = 4\pi \int_0^r \rho^2 d\rho$$

$$= 4\pi \, \frac{\rho^3}{3}\Big|_0^r = \frac{4}{3}\pi r^3. \qquad \blacksquare$$

Cylindrical Coord



$$\rho = \|(x,y,z)-(0,0,z)\|$$

$$= \|(x,y,0)\|$$

$$= \sqrt{x^2+y^2}$$

$$\boxed{x^2+y^2=\rho^2}$$

$$x^2+y^2+z^2 = \rho^2+z^2$$

$$\Phi^{-1} \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$\rho \geq 0$

$0 \leq \theta \leq 2\pi$

$-\infty < z < +\infty$

$$\det(\Phi^{-1})' = \det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\theta & \partial_z \end{array} \\ \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= +1 \det \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$= \rho$$

$$|\det(\Phi^{-1})'| = \rho.$$

Ex 4.7.4

#8 $\text{Vol}\left\{(x,y,z) : \boxed{z \geqslant x^2+y^2,} \; z \leqslant 18-x^2-y^2\right\}$

$z \leqslant 18-(x^2+y^2)$

$z \geqslant x^2+y^2$

How to draw these sets.

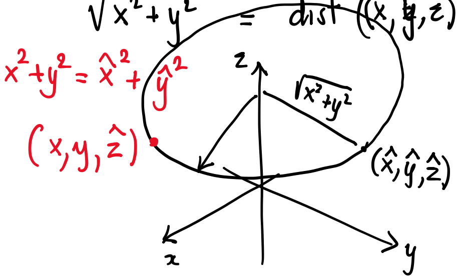
Things to know:

$$\sqrt{x^2+y^2+z^2} = \text{dist}\left((x,y,z), \vec{0}\right)$$

$$D = \left\{ f(x^2+y^2+z^2) \geqslant 0 \right\}$$

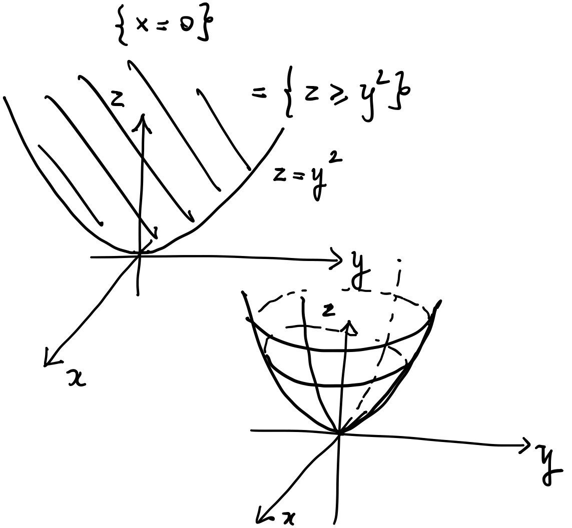
$$\text{Ex}: \left\{ x^2+y^2+z^2 \leqslant 1 \right\}$$

$$\sqrt{x^2+y^2} = \text{dist}\left((x,y,z), z\text{-axis}\right)$$



$$\hat{z} \geqslant \hat{x}^2+\hat{y}^2$$

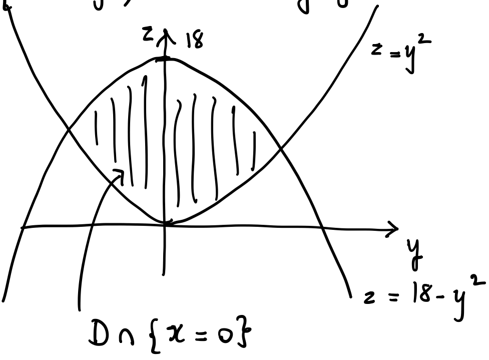
$$D \cap \text{plane } \underset{\shortparallel}{yz} = \left\{ z \geqslant x^2+y^2 \right\} \cap \left\{ x=0 \right\}$$

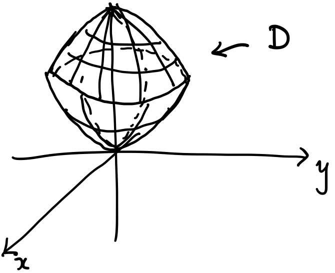


$$D = \{ z \geq x^2 + y^2, \quad z \leq 18 - (x^2+y^2) \}$$

$$D \cap \text{plane } yz = D \cap \{x = 0\}$$

$$= \{ z \geq y^2, \quad z \leq 18 - y^2 \}$$





$$\mathrm{Vol} = \int_{\substack{z \geqslant x^2+y^2 \\ z \leqslant 18-(x^2+y^2)}} 1 \quad dx\,dy\,dz$$

$$\underset{\text{cyl. coords}}{=} \int_{\substack{z \geqslant \rho^2 \\ z \leqslant 18-\rho^2 \\ 0 \leqslant \theta \leqslant 2\pi}} 1 \cdot \rho \quad d\rho\, d\theta\, dz$$

$$= \int_{\substack{z \geqslant \rho^2 \\ z \leqslant 18-\rho^2}} \int_0^{2\pi} \rho \, d\theta \quad d\rho\, dz$$

$$= 2\pi \int_{\substack{\boxed{z \geqslant \rho^2} \\ \boxed{z \leqslant 18-\rho^2}}} \rho \, d\rho\, dz$$

$$= 2\pi \int_0^3 \int_{\rho^2}^{18-\rho^2} \rho \, dz \quad d\rho$$

$$\boxed{\rho^2} \leqslant z \leqslant \boxed{18-\rho^2}$$

$$\rho^2 \leqslant 18-\rho^2 \iff 2\rho^2 \leqslant 18$$

$$\iff \underline{\rho^2 \leqslant 9}$$

$$= 2\pi \int_0^3 \rho\,(18-2\rho^2)\,d\rho \qquad \iff \boxed{0 \leqslant \rho \leqslant 3}$$

$$= 2\pi \int_0^3 18\rho - 2\rho^3 \, d\rho$$

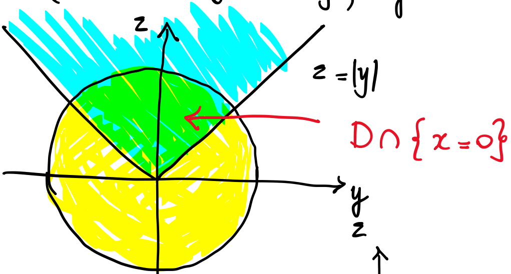
$$= 2\pi \left[\cancel{18}\,9\,\frac{\rho^2}{\cancel{2}}\Big|_0^3 - \cancel{2}\,\frac{\rho^4}{\cancel{4}\,2}\Big|_0^3 \right]$$

$$= 2\pi \left[81 - \frac{81}{2} \right] = 81\pi. \qquad \blacksquare$$

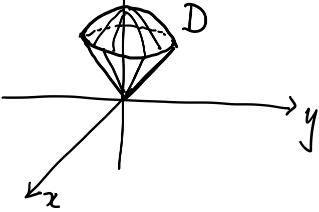
Do Ex 4.7.4 5, 6, 7, 3, 1, 2, 4

#7 Vol $\underbrace{\left\{ z \geqslant \sqrt{x^2+y^2},\; x^2+y^2+z^2 \leqslant 1 \right\}}_{D}$

$$D \cap \{x=0\} = \{ z \geqslant |y|,\; y^2+z^2 \leqslant 1 \}$$



$$\text{Vol } D = \int_D 1 \, dx\,dy\,dz$$



$$= \int_{\substack{z \geq \sqrt{x^2+y^2} \\ x^2+y^2+z^2 \leq 1}} 1 \quad dx\,dy\,dz$$

$$\underset{\text{cyl coords}}{=} \int_{\substack{z \geq \rho \\ \rho^2+z^2 \leq 1 \\ 0 \leq \theta \leq 2\pi}} 1 \quad \rho \quad d\rho\,d\theta\,dz$$

$$\overset{RF}{=} \int_{\substack{z \geq \rho \\ \rho^2+z^2 \leq 1}} \int_0^{2\pi} \rho \, d\theta \quad d\rho\,dz$$

$$= 2\pi \boxed{\int_{\substack{\boxed{z \geq \rho} \\ \boxed{\rho^2+z^2 \leq 1}}} \rho \quad d\rho\,dz}$$

$$= 2\pi \int_0^{1/\sqrt{2}} \int_\rho^{\sqrt{1-\rho^2}} \rho \, dz \quad d\rho$$

$$\boxed{z \geq \rho} \qquad z^2 \leq 1-\rho^2 \qquad\qquad 0 \leq \rho \leq \frac{1}{\sqrt{2}}$$

$$\underbrace{-\sqrt{1-\rho^2}} \leq z \leq \sqrt{1-\rho^2} \qquad\qquad \Updownarrow$$

$$\rho^2 \leq 1/2$$

$$\Uparrow$$

$$\rho \leq z \leq \underbrace{\sqrt{1-\rho^2}} \qquad 2\rho^2 \leq 1$$

$$\Updownarrow$$

$$\rho \leq \sqrt{1-\rho^2} \iff \rho^2 \leq 1-\rho^2$$

$$= 2\pi \int_0^{1/\sqrt{2}} \underbrace{\rho\left(\sqrt{1-\rho^2}\right.} - \left.\rho\right) \, d\rho$$

$$= 2\pi \int_0^{1/\sqrt{2}} \underbrace{(1-\rho^2)^{1/2}\rho} - \rho^2 \, d\rho$$

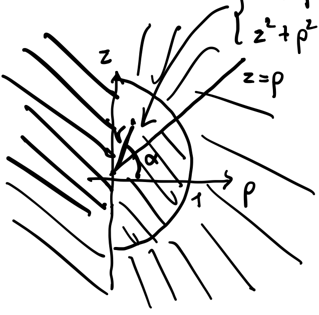
$$\partial_\rho \frac{(1-\rho^2)^{3/2}}{-3} = \frac{\cancel{-3}}{\cancel{2}}(1-\rho^2)^{1/2}(+\cancel{2}\rho)$$

$$= 2\pi\left[\left[\frac{(1-\rho^2)^{3/2}}{-3}\right]_{\rho=0}^{\rho=1/\sqrt{2}} - \frac{\rho^3}{3}\Big|_0^{1/\sqrt{2}}\right]$$

$= \ldots$

Smart sol: $\int \rho \, d\rho \, dz$

$$\begin{cases} z \geq \rho \\ z^2+\rho^2 \leq 1 \end{cases}$$



$$\rho = r\cos\alpha$$
$$z = r\sin\alpha$$

$$= \int r\cos\alpha \; r \; dr\,d\alpha$$

$\frac{\pi}{4} \leq \alpha \leq \pi/2$

$0 \leq r \leq 1$

= easier