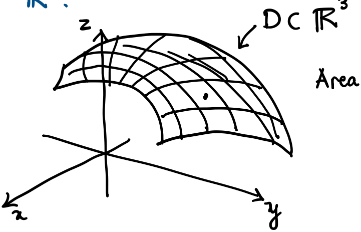
Monday, 13 March 2023 12:09

Surface Integrals

Pb: Compute the area of a surface in $\mathbb{R}^3$.



Area D.

$$\text{Vol } D = \int_D 1 \, dx\,dy\,dz = 0$$

and we cannot compute Area D =

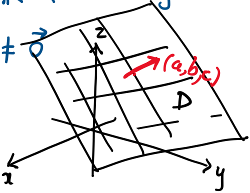
$$= \int_{D \subset \mathbb{R}^3} 1 \, dx\,dy$$

The first step is to give a look to possible ways to def a surface in $\mathbb{R}^3$. There're two principal ways:

Example (plane in $\mathbb{R}^3$)

$$D = \{(x,y,z) \in \mathbb{R}^3 : ax + by + cz = r\}$$

$$(a,b,c) \neq \vec{0}$$



This D can be also described in another way: since $(a,b,c) \neq \vec{0}$ at least one of $a,b,c \neq 0$. Let's say it is c

$$ax+by+\underbrace{cz}_{} = r$$

$$\cancel{c}z = \frac{r-(ax+by)}{c}$$

$$(x,y,z)\in D \iff \left(x,y,\frac{r-(ax+by)}{c}\right)$$

$$x,y\in\mathbb{R}^2$$

$$D = \left\{ \underbrace{\left(x,y,\frac{r-(ax+by)}{c}\right)}_{} : (x,y)\in\mathbb{R}^2 \right\}$$

$$= \Phi(x,y) \qquad \Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

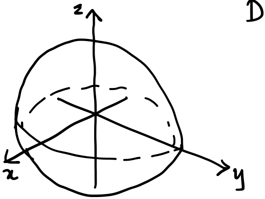
$$(x,y) \longmapsto \left(x,y,\frac{r-(ax+by)}{c}\right)$$

$$D = \boxed{\Phi}(\mathbb{R}^2).$$

↑

parametrization.

- (sphere of $\mathbb{R}^3$)



$$D = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=r^2\}$$

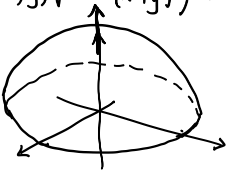
$$(x,y,z)\in D$$

$$\iff x^2+y^2+z^2=r^2$$

$$\iff z^2 = r^2-(x^2+y^2)$$

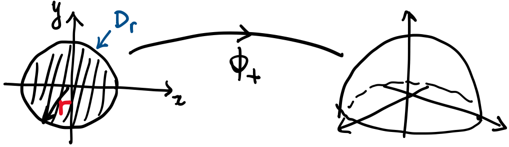
$$\Leftrightarrow \quad z = \pm\sqrt{r^2-(x^2+y^2)}$$

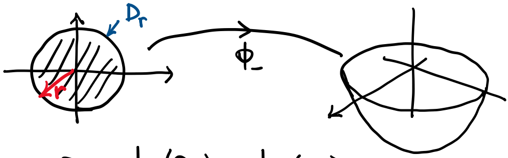
$$\Leftrightarrow \quad D = \Big\{ \overbrace{\left(x,y,\sqrt{r^2-(x^2+y^2)}\right)}^{\phi_+(x,y)} : x^2+y^2 \leq r^2 \Big\}$$



$$\cup \; \Big\{ \underbrace{\left(x,y,-\sqrt{r^2-(x^2+y^2)}\right)}_{\phi_-(x,y)} : x^2+y^2 \leq r^2 \Big\}$$

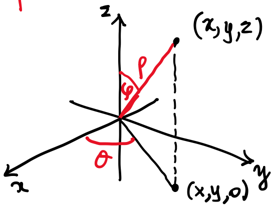
$$\phi_+, \phi_- : \underbrace{\boxed{\{(x,y) : x^2+y^2 \leq r^2\}}}_{\subset \mathbb{R}^2} \longrightarrow \mathbb{R}^3$$





$$D = \phi_+(D_r) \cup \phi_-(D_r)$$

An alternative parametrization uses spherical coords:



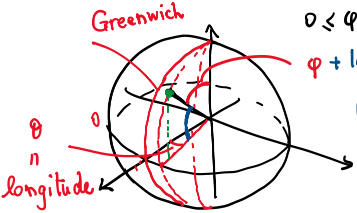
$$\begin{cases} x = \rho \sin\varphi \cos\vartheta \\ y = \rho \sin\varphi \sin\vartheta \\ z = \rho \cos\varphi \end{cases}$$

$\rho \geq 0, \quad 0 \leq \varphi \leq \pi$

$0 \leq \vartheta \leq 2\pi.$

So sphere

$$D = \{ \overbrace{(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)}^{\Phi(\varphi,\theta)} : 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi \}$$



φ + latitude $= \frac{\pi}{2}$

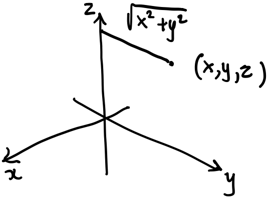
lat $= \frac{\pi}{2} - \varphi$.

$\uparrow$

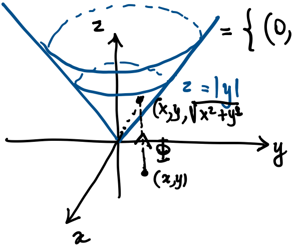
co-latitude

$$D = \Phi([0,\pi] \times [0, 2\pi])$$

Ex: $D = \{ (x,y,z) \in \mathbb{R}^3 : z = \underbrace{\sqrt{x^2+y^2}}_{} \}$



$D \cap$ plane $yz = D \cap \{x = 0\}$



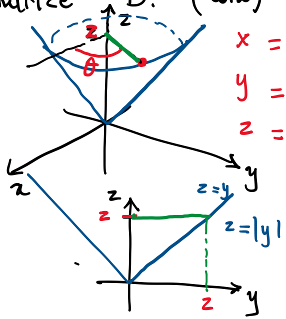
$= \{ (0,y,z) : z = \sqrt{y^2} = |y| \}$

$\Rightarrow$ D is a cone.

$$D = \{ \underbrace{(x,y,\sqrt{x^2+y^2})}_{\Phi(x,y)} : (x,y) \in \mathbb{R}^2 \}$$

$= \Phi(\mathbb{R}^2)$. Φ is a param of D

This is not the unique way we can paramtrize D. (cone)



$$x = \overset{z}{\overset{\|}{\bullet}} \cos\theta$$

$$y = \overset{z}{\overset{\|}{\bullet}} \sin\theta$$

$$z = z$$

$$D = \{ \underbrace{(z\cos\theta, z\sin\theta, z)}_{\Phi(z,\theta)} : z \geq 0, 0 \leq \theta \leq 2\pi \}$$

$$D = \Phi\left([0, +\infty[\times [0, 2\pi] \right).$$

Conclusion: A surface $D \subset \mathbb{R}^3$ can be defined into two principal ways:

- through an equation

$$D = \{ (x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0 \}$$

- through a parametrization

$$D = \Phi(E) \qquad \Phi = \Phi(u,v) : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

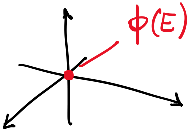
In general, without supplementary assumptions on Φ, $D = \Phi(E)$ is not a surface.

For example:

- (stupid ex.) $\Phi : E = \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

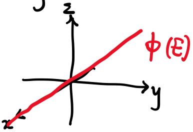
$$(u,v) \longmapsto (0,0,0)$$

$$\phi(E) = \{\phi(u,v) : (u,v) \in \mathbb{R}^2\}$$
$$= \{(0,0,0)\}$$



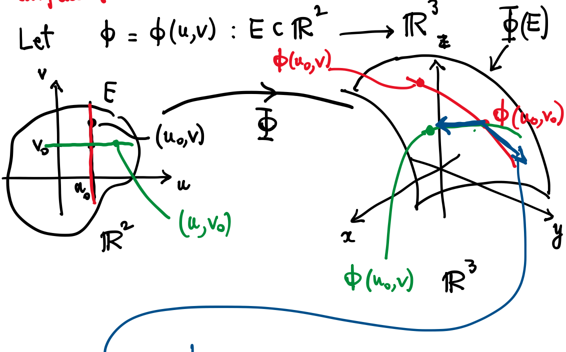
- $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
  $(u,v) \longmapsto (u,0,0)$

$$\phi(E) = \{(u,0,0) : u \in \mathbb{R}\} = x\text{-axis}$$



What condition on ϕ makes $\phi(E)$ a true surface?

Let $\phi = \phi(u,v) : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$



$$\frac{d}{dv}\phi(u_0,v) = \underline{\partial_v}\,\phi(u_0,v)$$

Rmk: $\phi(u,v) = (\phi_1(u,v), \phi_2(u,v), \phi_3(u,v))$

$$\partial_v \phi = (\partial_v \phi_1, \partial_v \phi_2, \partial_v \phi_3)$$

Similarly we define

$$\partial_u \phi = (\partial_u \phi_1, \partial_u \phi_2, \partial_u \phi_3)$$

Def: We say ϕ is an immersion on E if

$$\partial_u \phi(u,v),\ \partial_v \phi(u,v)$$

are linearly independent $\forall (u,v) \in E$.

Rmk: $\partial_u\phi, \partial_v\phi$ are lin indep $\Leftrightarrow$

$$\operatorname{rk}\underbrace{\begin{bmatrix}\partial_u\phi\\ \partial_v\phi\end{bmatrix}}_{2\times 3 \text{ matrix}} = 2$$

So ϕ is not an immersion $\Leftrightarrow$ there're pts where

$$\operatorname{rk}\begin{bmatrix}\partial_u\phi\\ \partial_v\phi\end{bmatrix} < 2$$

$\Leftrightarrow$ all 2×2 sub dets $= 0$. $\blacksquare$

Example:

$$D = \text{cone} = \left\{ (z\cos\theta, z\sin\theta, z)\ ;\ \begin{matrix} z \geq 0 \\ 0 \leq \theta \leq 2\pi \end{matrix} \right\}$$

Here $\phi(\underset{u}{z}, \underset{v}{\theta}) = (z\cos\theta, z\sin\theta, z)$

$$\partial_z \phi = (\cos\theta, \sin\theta, 1)$$

$$\partial_\theta \phi = (-z\sin\theta, z\cos\theta, 0)$$

$$\Rightarrow \phi \text{ is immersion} \Leftrightarrow \operatorname{rk}\begin{bmatrix}\partial_z\phi\\ \partial_\theta\phi\end{bmatrix} = 2$$

Now $\operatorname{rk}\begin{bmatrix}\partial_z\phi\\ \partial_\theta\phi\end{bmatrix} = \operatorname{rk}\begin{bmatrix}\cos\theta & \sin\theta & 1\\ -z\sin\theta & z\cos\theta & 0\end{bmatrix} < 2$

$$\Leftrightarrow \left\{\begin{array}{l} \det\begin{bmatrix} \cos\theta & \sin\theta \\ -z\sin\theta & z\cos\theta \end{bmatrix} = 0 \\ \det\begin{bmatrix} \cos\theta & 1 \\ -z\sin\theta & 0 \end{bmatrix} = 0 \\ \det\begin{bmatrix} \sin\theta & 1 \\ z\cos\theta & 0 \end{bmatrix} = 0 \end{array}\right. \quad \left\{\begin{array}{l} z(\cos^2\theta+\sin^2\theta)=0 \\ z\sin\theta = 0 \\ -z\cos\theta = 0 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} z=0 \\ 0=0 \\ 0=0 \end{array}\right. \quad \Leftrightarrow \quad z=0 \qquad \phi(0,\theta)=(0,0,0)$$

$\Rightarrow$ ϕ is an immersion on $D\setminus\{(0,0,0)\}$. ■

$$D = \{\underbrace{(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)}_{\phi(\varphi,\theta)}\}$$

(surface of sphere)

Here

$$\partial_\varphi \phi = (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ -r\sin\varphi)$$

$$\partial_\theta \phi = (-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0)$$

$$\operatorname{rk}\begin{bmatrix} \partial_\varphi \Phi \\ \partial_\theta \Phi \end{bmatrix} < 2 \Leftrightarrow \left\{\begin{array}{l} r^2\sin\varphi\cos\varphi\cos^2\theta + r^2\sin\varphi\cos\varphi\sin^2\theta = 0 \\ r^2\sin^2\varphi\sin\theta = 0 \\ r^2\sin^2\varphi\cos\theta = 0 \end{array}\right.$$

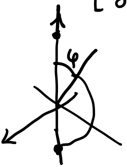
$$\Leftrightarrow \left\{\begin{array}{l} \cancel{r^2}\sin\varphi\cos\varphi = 0 \Leftrightarrow \sin 2\varphi = 0 \Leftrightarrow 2\varphi = 0,\pi,2\pi \\ \qquad\qquad\qquad\qquad\qquad\qquad \Leftrightarrow \varphi = 0,\pi/2,\pi \\ \cancel{r^2}\sin^2\varphi\sin\theta = 0\ . \\ \cancel{r^2}\sin^2\varphi\cos\theta = 0\ . \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \boxed{\varphi = 0,\pi} \\ 0=0 \\ 0=0 \end{array}\right. \vee \left\{\begin{array}{l} \varphi = \pi/2 \\ \cos\theta = 0 \\ \sin\theta = 0 \end{array}\right. \quad \text{imposs.}$$

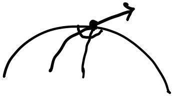
So $\operatorname{rk}\begin{bmatrix}\partial_\varphi \Phi \\ \partial_\theta \Phi\end{bmatrix} < 2 \iff \varphi = 0, \pi$

$\varphi = 0 \iff$ north pole

$\varphi = \pi \iff$ south pole



Conclusion: Φ is an immersion on the sphere except at N, S poles



■

The case when the surface can be represented as

$$\boxed{z = f(x,y)} \qquad (x,y) \in E$$

is a general important case:

$$D = \Big\{ \underbrace{(x, y, f(x,y))}_{\Phi(x,y)} \; : \; (x,y) \in E \Big\}$$

(standard param)

Φ is always an immersion, $\forall f$.

Indeed

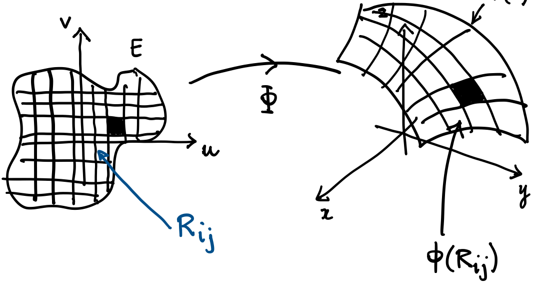
$$\begin{aligned} \partial_x \Phi &= (1, 0, \partial_x f) \\ \partial_y \Phi &= (0, 1, \partial_y f) \end{aligned} \quad \Rightarrow \operatorname{rk}\begin{bmatrix}\partial_x \Phi \\ \partial_y \Phi\end{bmatrix} = 2 \quad \forall (x,y).$$

Def: We call parametric surface a set

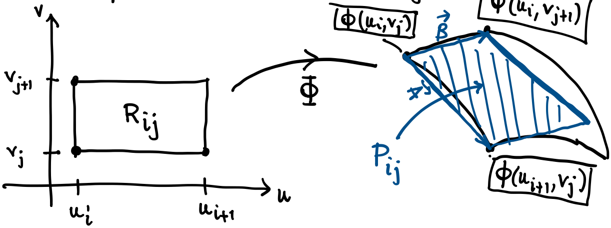
$$D = \Phi(E),$$

where Φ immersion on E.

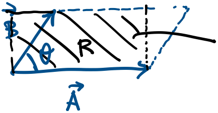
We see now a method to compute the area of a parametric surface:



$$\text{Area } \phi(E) = \sum \text{Area } \Phi(R_{ij})$$



$$\text{Area } \Phi(R_{ij}) \approx \text{Area } P_{ij}$$



$$\text{Area } P = \text{Area } R$$

$$= \|\vec{A}\| \|\vec{B}\| \sin\theta$$

$$= \|\vec{A} \wedge \vec{B}\|$$

$$\Rightarrow \quad \text{Area } P_{ij} = \|\vec{A} \wedge \vec{B}\|$$

In our case

$$\vec{A} = \phi(u_{i+1}, v_j) - \phi(u_i, v_j) \approx \overset{\text{vect}}{\partial_u \phi(u_i, v_j)} \overset{\text{scalar}}{(u_{i+1} - u_i)} = \partial_u \phi \cdot du_i$$

$\ulcorner f(b) - f(a) = f'(c)(b-a) \qquad c \in]a,b[\lrcorner$

scal

$$\vec{B} = \phi(u_i,v_{j+1}) - \phi(u_i,v_j) \approx \partial_v\phi(u_i,v_j)(v_{j+1}-v_j) = \partial_v\phi \cdot dv_j$$

vect

$$\Rightarrow \text{Area } P_{ij} = \| \partial_u\phi \, du_i \wedge \partial_v\phi \, dv_j \|$$

$$= du_i \, dv_j \, \| \partial_u\phi \wedge \partial_v\phi \| .$$

$$\Rightarrow \text{Area } \phi(E) = \sum \text{Area } \phi(R_{ij})$$

$$= \sum \| \partial_u\phi \wedge \partial_v\phi \| \, du_i \, dv_j$$

$$= \int_E \| \partial_u\phi \wedge \partial_v\phi \| \, du \, dv$$

$$\boxed{\text{Area } \Phi(E) = \int_E \underbrace{\| \partial_u\phi \wedge \partial_v\phi \|}_{} \, du \, dv} \longrightarrow \text{area element}$$

Ex 1 Area of surface of a sphere of radius r.

Sol: We use the param of the sphere with spherical coords:

$$\phi(\varphi,\theta) = (r\sin\varphi\cos\theta, \; r\sin\varphi\sin\theta, \; r\cos\varphi)$$

$$(\varphi,\theta) \in E = \{0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$$

$$\partial_\varphi \phi \wedge \partial_\theta \phi = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ & \partial_\varphi \phi & \\ & \partial_\theta \phi & \end{bmatrix}$$

$$= \det \begin{bmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{bmatrix}$$

$$= \left(r^2\sin^2\varphi\cos\theta,\ -\left(-r^2\sin^2\varphi\sin\theta\right),\ r^2\sin\varphi\cos\varphi\right)$$

$$\Rightarrow \|\partial_\varphi \phi \wedge \partial_\theta \phi\| = \sqrt{\underbrace{r^4\sin^4\varphi\cos^2\theta + r^4\sin^4\varphi\sin^2\theta}\ + r^4\sin^2\varphi\cos^2\varphi}$$

$$r^4\sin^4\varphi + r^4\sin^2\varphi\cos^2\varphi$$

$$r^4\sin^2\varphi\left(\underbrace{\sin^2\varphi + \cos^2\varphi}_{1}\right)$$

$$= \sqrt{r^4\sin^2\varphi} = r^2|\sin\varphi|$$

$$0 \leq \varphi \leq \pi \Rightarrow \sin\varphi \geq 0$$

$$= r^2\sin\varphi.$$

$$\text{Area}(\text{Sphere}(r)) = \int_{\substack{0 \leq \varphi \leq \pi \\ 0 \leq \theta \leq 2\pi}} r^2\sin\varphi \, d\varphi \, d\theta$$

$$\overset{RF}{=} r^2 \cdot \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \; d\theta$$

$$-\cos\varphi \Big|_0^{\pi} = -(-1-1) = +2$$

$$= 2r^2 \int_0^{2\pi} d\theta = 4\pi r^2 \qquad \blacksquare$$

In case $D = \{ z = f(x,y), \ (x,y) \in E \}$ we have a simplified formula for the area:

$$\text{Area } D = \int_E \sqrt{1 + \| \underset{\overset{\|}{(\partial_x f, \partial_y f)}}{\nabla f} \|^2} \; dx\, dy$$

Indeed, in this case there's a standard param:

$$\phi(x,y) = (x, y, f(x,y)) \qquad (x,y) \in E$$

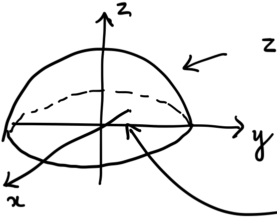
Then, the area element

$$\| \partial_x \phi \wedge \partial_y \phi \| = \left\| \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x f \\ 0 & 1 & \partial_y f \end{bmatrix} \right\|$$

$$= \| (-\partial_x f, -\partial_y f, 1) \|$$

$$= \sqrt{1 + \underbrace{(\partial_x f)^2 + (\partial_y f)^2}_{\|\nabla f\|^2}} \qquad \blacksquare$$

Let's redo the area of a sphere:



$$z = \underbrace{\sqrt{r^2 - (x^2+y^2)}}_{f(x,y)}$$

$$(x,y) \in E = \{x^2+y^2 \leqslant r^2\}$$

$$\Rightarrow \quad \text{Area}(\text{Sphere}(r)) = 2 \text{ Area})$$

$$= 2 \int_{x^2+y^2 \leqslant r^2} \sqrt{1 + \|\nabla f\|^2} \, dx\,dy$$

$$\nabla f = (\partial_x f, \partial_y f) \qquad f = \sqrt{r^2 - (x^2+y^2)}$$

$$= \left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{r^2-(x^2+y^2)}}, \frac{-\cancel{2}y}{\cancel{2}\sqrt{r^2-(x^2+y^2)}} \right)$$

$$\|\nabla f\|^2 = \frac{x^2}{r^2-(x^2+y^2)} + \frac{y^2}{r^2-(x^2+y^2)}$$

$$\sqrt{1+\|\nabla f\|^2} = \sqrt{\frac{r^2 - \cancel{(x^2+y^2)} + \cancel{x^2+y^2}}{r^2-(x^2+y^2)}}$$

$$= \frac{r}{\sqrt{r^2-(x^2+y^2)}}.$$

$$\Rightarrow \quad \text{Area}(\tfrac{1}{2}\text{ Sphere}) = \int_{x^2+y^2 \leqslant r^2} \frac{r}{\sqrt{r^2-(x^2+y^2)}} \, dx\,dy$$

$$= r \int_{\substack{\rho^2 \leq r^2 \\ 0 \leq \theta \leq 2\pi}} \frac{\cancel{r}}{\sqrt{r^2-\rho^2}} \, \rho \, d\rho \, d\theta$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\overset{RF}{=} -2\pi r \int_0^r \frac{-2\rho}{+2\sqrt{r^2-\rho^2}} \, d\rho$$

$$= -2\pi r \left[\sqrt{r^2-\rho^2} \right]_{\rho=0}^{\rho=r} = 2\pi r^2.$$

$$0 - r$$

$\Rightarrow$ Area Sphere$(r) = 4\pi r^2$ ■

Do 5.8.1, 5.8.2 (#2 difficult)