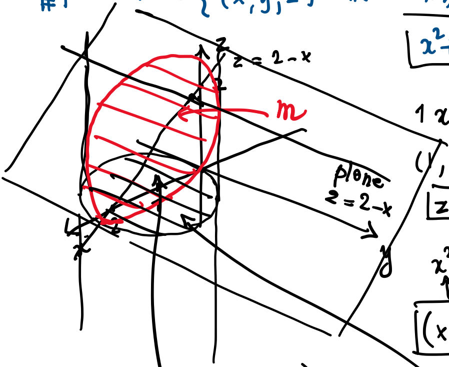
Tuesday, 14 March 2023 12:10

Ex 5.8.2 Compute the area of

#1 $\mathcal{M} = \{(x,y,z)\in\mathbb{R}^3 : x \geqslant 0, z \geqslant 0, \boxed{x+z=2}, \boxed{x^2+y^2-2x\leqslant 0}\}$



$1x + 0y + 1z = 2$

$(1,0,1)$

$\boxed{z = 2-x} = f(x,y)$

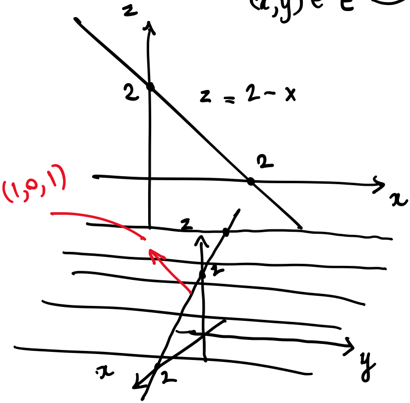
$x^2+y^2-2x+1-1\leqslant 0$

$\boxed{(x-1)^2+y^2\leqslant 1}$

$(x, y, f(x,y))$

$(x,y,2-x) =: \phi(x,y)$

$(x,y)\in E = \{(x-1)^2+y^2\leqslant 1\}$



Area $\mathcal{M} = \int_E \|\partial_x\phi \wedge \partial_y\phi\| \, dx\,dy$

$$= \int_E \sqrt{1 + \underbrace{\|\nabla f\|^2}_{=1}} \; dx\,dy$$

where $f(x,y) = 2 - x$

$\nabla f = (\partial_x f, \partial_y f) = (-1, 0)$

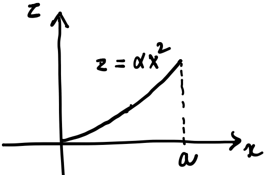
$\|\nabla f\|^2 = 1$

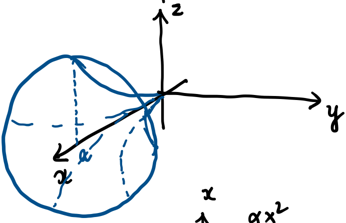
$$= \int_E \sqrt{2} \; dx\,dy$$

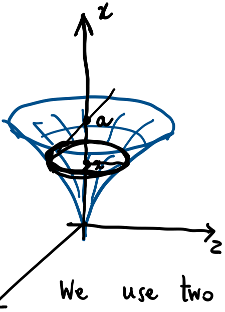
$$= \sqrt{2} \underbrace{\int_E 1 \; dx\,dy}_{\text{area of } E \text{ in } \mathbb{R}^2 \; = \pi \cdot 1^2 = \pi} = \sqrt{2}\,\pi .$$

∎

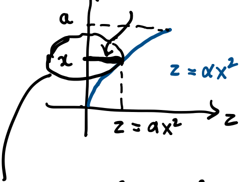
#3 Area m, m rotation surface around x-axis of $f(x) = \alpha x^2$ $x \in [0,a]$, $a, \alpha > 0$.







We use two coords, x and



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$$(x, \alpha x^2\cos\theta, \alpha x^2\sin\theta) =: \Phi(x,\theta)$$

$$\boxed{0 \leqslant x \leqslant a, \quad 0 \leqslant \theta \leqslant 2\pi.}$$

$$\text{Area } \mathfrak{M} = \int_{\substack{0 \leqslant x \leqslant a \\ 0 \leqslant \theta \leqslant 2\pi}} \| \partial_x \phi \wedge \partial_\theta \phi \| \, dx\, d\theta$$

$$\partial_x \phi \wedge \partial_\theta \phi = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2\alpha x\cos\theta & 2\alpha x\sin\theta \\ 0 & -\alpha x^2\sin\theta & \alpha x^2\cos\theta \end{bmatrix}$$

$$= \left(2\alpha^2 x^3 \overset{\overset{1}{\shortparallel}}{(\cos^2\theta + \sin^2\theta)}, -\alpha x^2\cos\theta, -\alpha x^2\sin\theta\right)$$

$$= \left(2\alpha^2 x^3, -\alpha x^2\cos\theta, -\alpha x^2\sin\theta\right)$$

$$\|\partial_x\phi \wedge \partial_\theta\phi\| = \sqrt{4\alpha^4 x^6 + \alpha^2 x^4 \underset{\underset{1}{\shortparallel}}{(\cos^2\theta + \sin^2\theta)}}$$

$$= \sqrt{\underline{4\alpha^4 x^6} + \underline{\alpha^2 x^4}}$$

$$= \alpha x^2 \sqrt{1 + 4\alpha^2 x^2}$$

$$\Rightarrow \quad \text{Area } \mathfrak{M} = \int_{\substack{0 \leqslant x \leqslant a \\ 0 \leqslant \theta \leqslant 2\pi}} \underline{\alpha x^2 \sqrt{1 + 4\alpha^2 x^2}} \, dx\, d\theta$$

$$\overset{RF}{=} \int_0^a \alpha x^2 \sqrt{1+4\alpha^2 x^2} \int_0^{2\pi} 1 \, d\theta \quad dx$$

$$= 2\pi\alpha \int_0^a x^2 \sqrt{1+4\alpha^2 x^2} \; dx \leftarrow$$

$$(2\alpha x)^2 \qquad u = Sh^{-1}(2\alpha x)$$

$$Ch^2 - Sh^2 = 1 \qquad 2\alpha x = Sh\, u$$

$$Ch^2 = 1 + Sh^2 \qquad x = \frac{1}{2\alpha} Sh\, u$$

$$0 \leq x \leq a \Rightarrow 0 \leq u \leq Sh^{-1}(2\alpha a) \quad dx = \underbrace{\frac{1}{2\alpha} Ch\, u \, du}$$

$$= \cancel{2}\pi\cancel{\alpha} \int_0^{Sh^{-1}(2\alpha a)} \frac{1}{4\alpha^2} Sh^2 u \cdot Ch\, u \frac{1}{\cancel{2\alpha}} Ch u \, du .$$

$$\sqrt{1+(2\alpha x)^2} = \sqrt{1 + Sh^2 u} = \sqrt{Ch^2 u} = |Ch\, u|$$

$$= Ch\, u$$

$$= \frac{\pi}{4\alpha^2} \int_0^{Sh^{-1}(2\alpha a)} Sh^2 u \; Ch^2 u \; du$$

$$\frac{1}{4} \left(Sh^2 2u \right)$$

$$Sh\, 2u = 2\, Sh\, u\, Ch\, u$$

$$\int \underline{\underline{Sh^2(2u)\, du}} = \int Sh(2u)\, Sh(2u)$$

$$\| \qquad \left(\frac{Ch\, 2u}{2}\right)'$$

$$= \frac{1}{2} Sh\, 2u \cdot Ch\, 2u$$

$$-\int \cancel{2}\,\mathrm{Ch}\,2u \qquad \frac{\mathrm{Ch}\,2u}{\cancel{2}}$$

$$= \frac{1}{2}\,\mathrm{Sh}\,2u\,\mathrm{Ch}\,2u - \int \mathrm{Ch}^2 2u \, du$$

$$= 1 + \mathrm{Sh}^2(2u)$$

$$= \frac{1}{2}\,\mathrm{Sh}(2u)\,\mathrm{Ch}(2u) - u - \int \mathrm{Sh}^2(2u)$$

$$\cancel{2}\int \mathrm{Sh}^2(2u)\,du = \frac{1}{4}\,\mathrm{Sh}(2u)\,\mathrm{Ch}(2u) - \frac{u}{2}.$$

■

Do 5.8.2 #4.

M Rotation around z-axis of

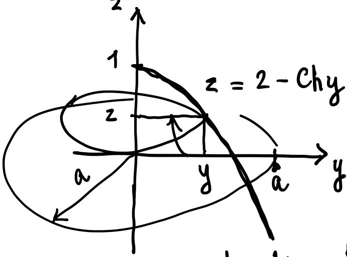
$$z = f(y) = 2 - \mathrm{Ch}\,y \qquad y \in [0,a]$$

$a > 0$

Area of M

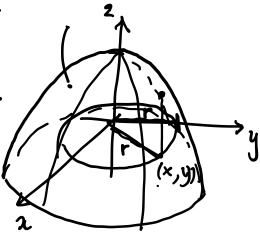
Step 1: plot $z = 2 - \mathrm{Ch}\,y$ $\quad y \in [0,a]$

in plane yz.



Step 2: parametrization

$(x,y,z) =$



$$z = 2 - \mathrm{Ch}\, y$$

$$\mathrm{Ch}\, y = 2 - z$$

$$y = \underline{\mathrm{Ch}^{-1}(2-z)}$$

$$z = 2 - \mathrm{Ch}\, r$$

$$z = 2 - \mathrm{Ch}\sqrt{x^2+y^2}$$

$$\left(\mathrm{Ch}^{-1}(2-z)\cos\theta,\ \mathrm{Ch}^{-1}(2-z)\sin\theta,\ z\right)$$

$$z = 2 - \mathrm{Ch}\sqrt{x^2+y^2} = f(x,y)$$

$$x^2+y^2 \leqslant a^2$$

$$\text{Area} = \int_{x^2+y^2\leqslant a^2} \sqrt{1 + \|\nabla f\|^2}\ dx\,dy$$

$$\nabla f = \left(-\mathrm{Sh}\sqrt{x^2+y^2}\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}},\ -\mathrm{Sh}\sqrt{x^2+y^2}\cdot\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\|\nabla f\|^2 = \mathrm{Sh}^2\sqrt{x^2+y^2}\cdot\frac{x^2}{x^2+y^2} + \mathrm{Sh}^2\sqrt{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}$$

$$= \mathrm{Sh}^2\sqrt{x^2+y^2}$$

$$= \int_{x^2+y^2\leqslant a^2} \sqrt{\underbrace{1 + \mathrm{Sh}^2\sqrt{x^2+y^2}}_{\mathrm{Ch}^2\sqrt{x^2+y^2}}}\ dx\,dy$$

$$= \int_{x^2+y^2\leqslant a^2} \mathrm{Ch}\sqrt{x^2+y^2}\ dx\,dy$$

$$\underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \int_{\substack{0\leqslant\rho\leqslant a\\ 0\leqslant\theta\leqslant 2\pi}} \mathrm{Ch}\,\rho\ \rho\ d\rho\, d\theta$$

$$\overset{RF}{=} 2\pi\int_0^a \rho\,\mathrm{Ch}\,\rho\ d\rho = \cdots \qquad \blacksquare$$

Surface Integral

Def: Given a parametric surface

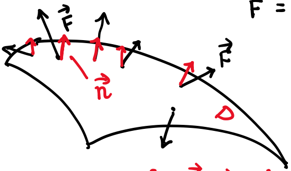
$D = \underline{\Phi}(E)$ $\quad \underline{\Phi} = \underline{\Phi}(u,v) : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

a function $f = f(x,y,z)$, we set

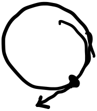
$$\int_D f \, d\sigma := \int_E f(\phi(u,v)) \, \|\partial_u \phi \wedge \partial_v \phi\| \, du dv.$$

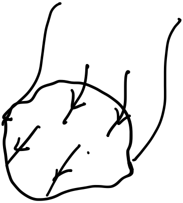
Fluxes

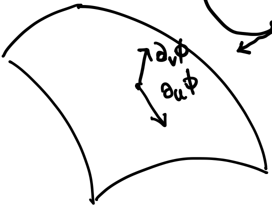
$\vec{F} = \vec{F}(x,y,z)$



$$\sum_{P \in \text{surface}} \vec{F} \cdot \vec{n} = \int_D \vec{F} \cdot \vec{n} \, d\sigma$$







$$\underbrace{\partial_u \phi \wedge \partial_v \phi}_{\text{vector} \perp \text{to surface}} \perp \partial_u \phi, \partial_v \phi$$

$$\vec{n} := \frac{\partial_u \phi \wedge \partial_v \phi}{\|\partial_u \phi \wedge \partial_v \phi\|}$$

$$\left(\|\vec{n}\| = \left\| \frac{\partial_u \phi \wedge \partial_v \phi}{\underbrace{\|\partial_u \phi \wedge \partial_v \phi\|}_{R}} \right\| = \frac{1}{\|\partial_u \phi \wedge \partial_v \phi\|} \cdot \|\partial_u \phi \wedge \partial_v \phi\| = 1 \right)$$

Def: Given $\vec{F} = \vec{F}(x,y,z) \in \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ ($\vec{F}$ is called vector field), a param. surf $D = \phi(E)$, $\phi : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, we call flux of $\vec{F}$ through D the quantity

$$\int_D \vec{F} \cdot \vec{n}_\phi \, d\sigma$$

where $\vec{n}_\phi = \dfrac{\partial_u \phi \wedge \partial_v \phi}{\| \partial_u \phi \wedge \partial_v \phi \|}$.

We notice that

$$\int_D \underbrace{\boxed{\vec{F} \cdot \vec{n}_\phi}}_{f} \, d\sigma = \int_E f(\phi(u,v)) \, \| \partial_u \phi \wedge \partial_v \phi \| \, du\,dv$$

$$= \int_E \boxed{\vec{F} \cdot \frac{\partial_u \phi \wedge \partial_v \phi}{\cancel{\| \partial_u \phi \wedge \partial_v \phi \|}}} \cdot \cancel{\| \partial_u \phi \wedge \partial_v \phi \|} \, du\,dv$$

$$= \int_E \underline{\vec{F} \cdot (\partial_u \phi \wedge \partial_v \phi)} \, du\,dv$$

$\vec{F} = (F_1, F_2, F_3)$

$\phi = (\phi_1, \phi_2, \phi_3)$

$$\partial_u \phi \wedge \partial_v \phi = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_u \phi_1 & \partial_u \phi_2 & \partial_u \phi_3 \\ \partial_v \phi_1 & \partial_v \phi_2 & \partial_v \phi_3 \end{bmatrix}$$

$$= \left(\underbrace{\det \begin{bmatrix} \partial_u \phi_2 & \partial_u \phi_3 \\ \partial_v \phi_2 & \partial_v \phi_3 \end{bmatrix}}_{\det_1}, \ominus \underbrace{\det \begin{bmatrix} \partial_u \phi_1 & \partial_u \phi_3 \\ \partial_v \phi_1 & \partial_v \phi_3 \end{bmatrix}}_{\det_2}, \underbrace{\det \begin{bmatrix} \partial_u \phi_1 & \partial_u \phi_2 \\ \partial_v \phi_1 & \partial_v \phi_2 \end{bmatrix}}_{\det_3} \right)$$

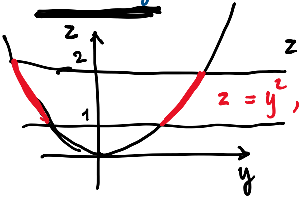
$$\vec{F} \cdot \partial_u \phi \wedge \partial_v \phi = F_1 \cdot \det_1 - F_2 \det_2 + F_3 \det_3$$

$$= \det \begin{bmatrix} F_1 & F_2 & F_3 \\ \partial_u \phi_1 & \partial_u \phi_2 & \partial_u \phi_3 \\ \partial_v \phi_1 & \partial_v \phi_2 & \partial_v \phi_3 \end{bmatrix}$$

Prop:

$$\int_D \underline{\vec{F} \cdot \vec{n}_\phi} \, d\sigma = \int_E \det \begin{bmatrix} \vec{F} \\ \partial_u \phi \\ \partial_v \phi \end{bmatrix} du \, dv.$$

Ex 5.4.2 Compute the flux of $\vec{F} = (x,y,z)$ ($\vec{F}(x,y,z) = \underline{(x,y,z)}$) through the paraboloid $\underline{z = x^2 + y^2}$ with $1 \leqslant z \leqslant 2$.



$z = y^2, \quad 1 \leqslant z \leqslant 2$

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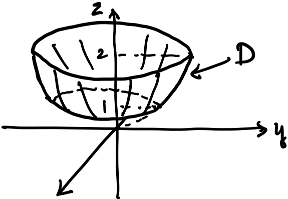


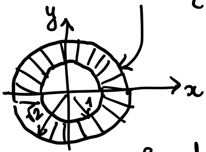
Let's write a param for D :

$(x,y,z) \in D \Leftrightarrow$ x

$\boxed{z = x^2+y^2, \quad 1 \le z \le 2} \Rightarrow 1 \le x^2+y^2 \le 2$

$\Leftrightarrow (x,y,z) = (x, y, \underline{x^2+y^2}) = \phi(x,y)$

$E = \{(x,y) : 1 \le x^2+y^2 \le 2\}$



$$\Rightarrow \int_D \vec F \cdot \vec n_\phi \, d\sigma = \int_E \det \begin{bmatrix} \vec F \\ \partial_x \phi \\ \partial_y \phi \end{bmatrix} dx\,dy$$

$$= \int_{1 \le x^2+y^2 \le 2} \det \begin{bmatrix} x & y & \overset{z}{\overset{\shortparallel}{x^2+y^2}} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix} dx\,dy$$

$$= \int_{1 \le x^2+y^2 \le 2} \left[\underline{x \cdot (-2x)} - 1 \cdot \left(2y^2 - \underline{(x^2+y^2)}\right)\right] dx\,dy$$

$$= \int_{1 \le x^2+y^2 \le 2} - x^2 - y^2 \; dx\,dy$$

ch. of var

$$\underset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{=} - \int_{\substack{1 \le \rho^2 \le 2 \\ 0 \le \theta \le 2\pi}} \rho^2 \cdot \rho \, d\rho \, d\theta$$

$$\overset{RF}{=} -2\pi \int_1^{\sqrt{2}} \rho^3 \, d\rho$$

$$= -\frac{2\pi}{4}\left[\frac{\rho^4}{\cancel{4}}\right]_1^{\sqrt{2}} = -\frac{\pi}{2}(4-1)$$

$$= -\frac{3}{2}\pi. \quad \blacksquare$$

Rmk In general, a flux depends on the param., however, by changing param. the unique change is (at most) the sign of flux.

$$\phi = \phi(u,v)$$

$$\psi = \psi(\underset{\uparrow}{u},v) = \underset{\smile}{\phi}(v,\underset{\smile}{u})$$

$$\vec{n}_\phi = \frac{\partial_u\phi \wedge \partial_v\phi}{\| \quad /\!/ \quad \|}$$

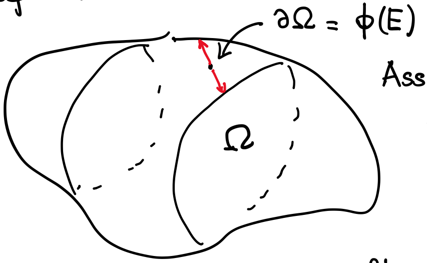
$$\vec{n}_\psi = \frac{\partial_u\psi \wedge \partial_v\psi}{\| \quad /\!/ \quad \|} = \frac{\partial_v\phi \wedge \partial_u\phi}{\| \quad /\!/ \quad \|}$$

$$= -\vec{n}_\phi$$

$$\int_D \vec{F}\cdot\vec{n}_\psi = -\int_D \vec{F}\cdot\vec{n}_\phi \qquad \blacksquare$$

Outward Flux

Let $\Omega \subset \mathbb{R}^3$ such that $\partial\Omega$ (boundary of Ω be a parametric surface



Assume $\vec{n}_\Phi$ points out of Ω

In this case a flux $\int_{\partial\Omega} \vec{F}\cdot\vec{n}_\phi$ is called outward flux.

Example: $\Omega = \{x^2+y^2+z^2 \le r^2\}$

$\partial\Omega = \{x^2+y^2+z^2 = r^2\}$

Thm (divergence) If $\Omega \subset \mathbb{R}^3$ is closed and bounded s.t. $\partial\Omega = \phi(E)$ with $\vec{n}_\phi$ pointing out of Ω $\Rightarrow$

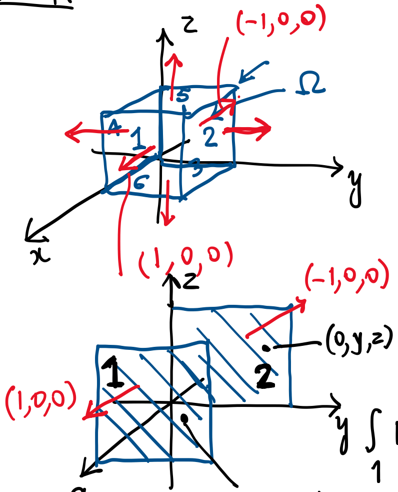
$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_\phi \, d\sigma = \int_{\Omega} \operatorname{div} \vec{F} \, dxdydz$$

where $\operatorname{div} \vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$

$\vec{F} = (F_1, F_2, F_3)$ $\qquad = \underbrace{(\partial_x, \partial_y, \partial_z)}_{\nabla} \underbrace{(F_1, F_2, F_3)}_{\vec{F}}$

$\operatorname{div} \vec{F} = $ "$\nabla \cdot \vec{F}$"

Proof: Ω = cube



$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\sigma = \boxed{\int_1 + \int_2} + \int_3 + \int_4 + \int_5 + \int_6$$

$$\int_1 \vec{F}\cdot\underset{(1,0,0)}{\vec{n}}\, d\sigma = \int_1 F_1\, d\sigma$$

$\phi(y,z) := (1,y,z)$

$0 \leq y \leq 1$

$0 \leq z \leq 1$

$$\boxed{= \int_{0\leq y,z\leq 1} F_1(1,y,z)\, dy dz}$$

$$\int_2 \vec{F}\cdot\underset{\underset{(-1,0,0)}{\shortparallel}}{\vec{n}}\, d\sigma = \int_2 -F_1\, d\sigma$$

$$\boxed{= \int_{0\leq y,z\leq 1} -F_1(0,y,z)\, dy dz}$$

$$\int_1 \vec{F}\cdot\vec{n} + \int_2 \vec{F}\cdot\vec{n} = \int_{0\leq y,z\leq 1} \underbrace{F_1(1,y,z) - F_1(0,y,z)}\, dy dz$$

$$\downarrow$$

$$F_1(1,y,z) - F_1(0,y,z) = F_1(1) - F_1(0)$$

$$= \boxed{\int_0^1 \partial_x F_1(x,y,z)\,dx}$$

$$\left(f(b) - f(a) = \int_a^b f'(x)\,dx \right)$$

$$\Rightarrow \int_1 \vec{F}\cdot\vec{n} + \int_2 \vec{F}\cdot\vec{n} = \int_{0\le y,z\le 1} \left(\int_0^1 \partial_x F_1(x,y,z)\,dx \right) dy\,dz$$

$$\overset{RF}{=} \int_{\underbrace{0\le x,y,z\le 1}_{\text{cube}}} \partial_x F_1(x,y,z)\,dx\,dy\,dz$$

$$= \int_\Omega \partial_x F_1 \,dx\,dy\,dz$$

$$\int_1 \vec{F}\cdot\vec{n} + \int_2 \vec{F}\cdot\vec{n} = \int_\Omega \partial_x F_1$$

$$\int_3 \vec{F}\cdot\vec{n} + \int_4 \vec{F}\cdot\vec{n} = \int_\Omega \partial_y F_2$$

$$\int_5 \vec{F}\cdot\vec{n} + \int_6 \vec{F}\cdot\vec{n} = \int_\Omega \partial_z F_3$$

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n} = \int_\Omega \underbrace{\left(\partial_x F_1 + \partial_y F_2 + \partial_z F_3 \right)}_{\operatorname{div}\vec{F}} \quad dx\,dy\,dz$$

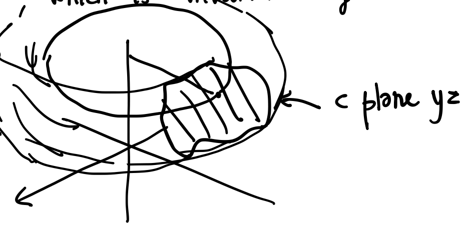
Ex 5.5.3

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 \le z \le 1+\sqrt{1-(x^2+y^2)}\}$$

Compute outward flux of $\vec{F}(x,y,z) = (x,y,x^2+y^2)$

Sol: (Not needed to compute the flux)

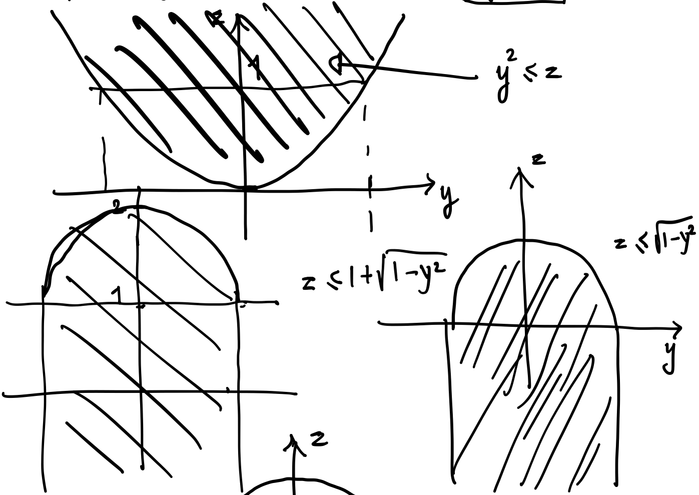
Let's draw Ω: since Ω depends on (x,y) through x^2+y^2, which is invariant by rots around z-axis,

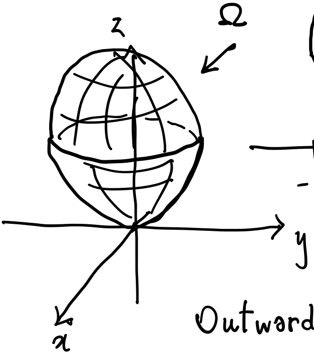


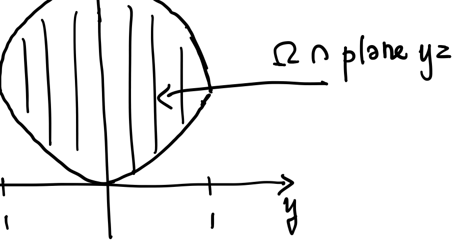
$\Downarrow$

Ω is invariant by rots around z-axis.

$$\Omega \cap \text{plane } yz = \Omega \cap \{x=0\} = \{\boxed{y^2 \le z} \le 1+\sqrt{1-y^2}\}$$







$$\text{Outward flux} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}$$

$$\overset{\text{div thm}}{\Rightarrow} \int_{\Omega} \operatorname{div} \vec{F} \; dx\,dy\,dz$$

$$\operatorname{div} \vec{F} = \partial_x x + \partial_y y + \partial_z (x^2+y^2)$$

$$\vec{F} = (x, y, x^2+y^2)$$

$$= 1 + 1 + 0 = 2$$

$$= \int_{\Omega} 2 \; dx\,dy\,dz$$

$$= 2 \int_{x^2+y^2 \leq z \leq 1+\sqrt{1-(x^2+y^2)}} 1 \qquad dx\,dy\,dz$$

$$\overset{RF}{=} 2 \int_{x^2+y^2\leq 1} \int_{x^2+y^2}^{1+\sqrt{1-(x^2+y^2)}} 1 \quad dz \quad dx\,dy$$

$x^2+y^2 \leq 1$

$$x^2+y^2 \le 1+\sqrt{1-(x^2+y^2)}$$

$$= 2\int_{x^2+y^2\le 1}\left(1+\sqrt{1-(x^2+y^2)}-(x^2+y^2)\right)dxdy$$

$$\underset{\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}}{=} 2\int_{\substack{0\le\rho\le1\\0\le\theta\le2\pi}}\left(1+\sqrt{1-\rho^2}-\rho^2\right)\rho\,d\rho d\theta$$

$$\overset{RF}{=} 4\pi\int_0^1 \underbrace{\rho}_{}-\rho^3+\overset{\rho(1-\rho^2)^{1/2}}{\rho\sqrt{1-\rho^2}}\,d\rho$$

$$=4\pi\left[\left.\frac{\rho^2}{2}\right|_0^1-\left.\frac{\rho^4}{4}\right|_0^1+\left.\frac{(1-\rho^2)^{3/2}}{-3}\right]_0^1\right]$$

$$\partial_\rho\frac{(1-\rho^2)^{3/2}}{-3}=\frac{\cancel{3}}{\cancel{2}}(1-\rho^2)^{1/2}(\cancel{-2}\rho)$$

$$=4\pi\left[\frac{1}{2}-\frac{1}{4}-\frac{1}{3}(0-1)\right].\qquad\blacksquare$$

Do 5.8.4/6.