Wednesday, 15 March 2023 13:42

Ex 5.8.4

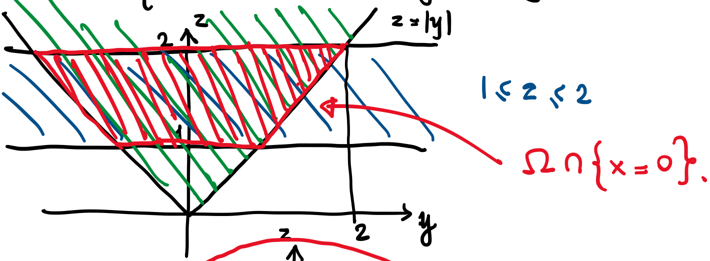
$$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : \ 1 \leq z \leq 2, \ \sqrt{x^2+y^2} \leq z \}$$

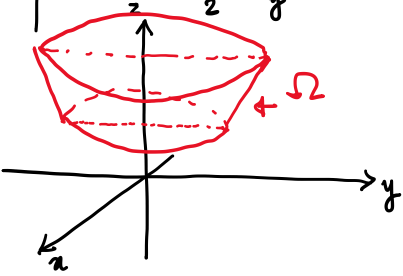
Q1 Draw Ω

Q2 Outward flux of $\vec{F}(x,y,z) = (x^3, 0, z^2)$ by Ω.

Sol: Q1 $\Omega \cap$ plane yz $= \Omega \cap \{x=0\}$

$= \{ 1 \leq z \leq 2, \quad z \geq \sqrt{y^2} = |y| \}$





Q2. To compute the outward flux we apply the divergence thm:

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\; d\sigma = \int_{\Omega} \operatorname{div}\vec{F}\; dx\,dy\,dz$$

$$\operatorname{div}\vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3 \;\Big|\; \vec{F} = (F_1, F_2, F_3)$$

In our case $\vec{F} = (x^3, 0, z^2)$

$$\operatorname{div}\vec{F} = \partial_x (x^3) + \partial_y 0 + \partial_z (z^2)$$

$$= 3x^2 + 2z.$$

$$\Rightarrow \int_{\partial\Omega} \vec{F}\cdot\vec{n}\; d\sigma = \int_{\Omega} (3x^2 + 2z)\; dx\,dy\,dz$$

$$= \int_{\substack{1\le z\le 2 \\ z \geq \sqrt{x^2+y^2}}} (3x^2 + 2z)\; dx\,dy\,dz$$

$$\overset{RF}{=} \cancel{\int_{x^2+y^2\le \boxed{z^2}} \int_1^2 f \; dz}$$

$$= \int_1^2 \Big(\int_{\sqrt{x^2+y^2}\le z} 3x^2 + \underline{2z}\; dxdy \Big)\; dz$$

$$= \int_1^2 \Big(\underbrace{\int_{\sqrt{x^2+y^2}\le z} 3x^2\; dxdy}_{} + \underline{2z} \underbrace{\int_{\sqrt{x^2+y^2}\le z} 1\; dxdy}_{= \underline{\pi z^2}} \Big)\; dz \underset{(xx)}{=}$$

$$\boxed{\int_{\sqrt{x^2+y^2}\le 2} 3x^2\,dx\,dy} =$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$= \int_{\substack{0\le\rho\le 2\\ 0\le\theta\le 2\pi}} 3\rho^2\cos^2\theta\,\boxed{\rho}\,d\rho\,d\theta$$

$$\overset{\underline{\text{RF}}}{=} \int_0^2 \left(\int_0^{2\pi} 3\rho^3\cos^2\theta\,d\theta\right) d\rho$$

$$= \int_0^2 3\rho^3 \int_0^{2\pi} \underset{\pi}{\underset{\shortparallel}{\cos^2\theta\,d\theta}}\;d\rho \underset{(*)}{=}$$

$$\int_0^{2\pi}\cos^2\theta = \int_0^{2\pi}\cos\theta\cdot(\sin\theta)'$$

$$= \overset{0}{\overset{\shortparallel}{\left[\cos\theta\sin\theta\right]}}_0^{2\pi} - \int_0^{2\pi} -\sin^2\theta$$

$$= +\int_0^{2\pi} \underset{1-\cos^2\theta}{\underset{\shortparallel}{\sin^2\theta}}$$

$$= 2\pi - \int_0^{2\pi}\cos^2\theta$$

$$\cancel{2}\int_0^{2\pi}\cos^2\theta = \cancel{2}\pi$$

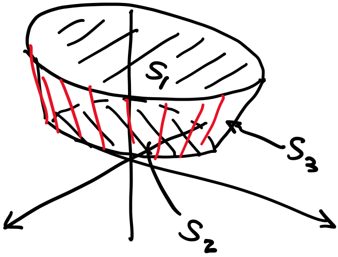
$$\underset{(*)}{=} 3\pi\int_0^2 \rho^3\,d\rho = 3\pi\left[\frac{\rho^4}{4}\right]_0^2$$

$$= \frac{3}{4}\pi z^4.$$

$$\overset{(**)}{=} \int_1^2 \left(\frac{3}{4}\pi z^4 + 2\pi z^3 \right) dz$$

$$= \pi\left(\frac{3}{4} \frac{z^5}{5}\Big|_1^2 - 2\frac{z^4}{4}\Big|_1^2 \right) = \dots$$

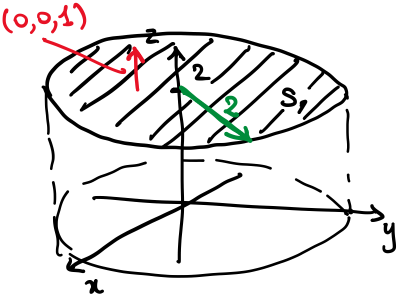
Q3. Det. components of outward flux on each of components of $\partial\Omega$.



Goal: compute

$$\int_{S_{1,2,3}} \vec{F}\cdot\underset{\substack{\uparrow\\ \text{outward}}}{\vec{n}}\, d\sigma$$

$$\int_{\partial\Omega} = \int_{S_1} + \int_{S_2} + \int_{S_3}$$



$$\int_{S_1} \vec{F}\cdot\underset{\substack{\downarrow\\ \text{outward}}}{\vec{n}}\, d\sigma$$

- $z = f(x,y) = 2$ $\quad (x,y) \in \underbrace{\{x^2+y^2 \leq 4\}}_{E}$

$\phi(x,y) = (x,y,2)$

$$\vec{n}_\phi = \frac{\partial_x \phi \wedge \partial_y \phi}{\| \quad // \quad \|}$$

$\partial_x \phi = (1,0,0)$

$\partial_y \phi = ($

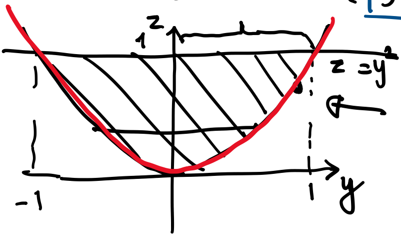
Ex 5.8.6

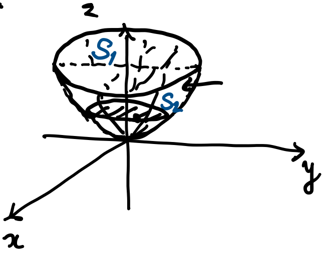
$\Omega = \{(x,y,z) : x^2+y^2 \leq z \leq 1\}$

Q1 Draw Ω.

Ω is invariant by rots around z-axis.

$\Omega \cap \{x=0\} = \{\boxed{y^2 \leq z} \leq 1\}$





Q2 Area$(\partial\Omega)$.

$$\text{Area}(\partial\Omega) = \underset{\pi\cdot 1^2}{\text{Area } S_1} + \text{Area } S_2$$

$$S_2 = \left\{(x,y,z)\in\mathbb{R}^3 : \underline{z = \overset{f(x,y)}{x^2+y^2}},\ \underline{x^2+y^2\leq 1}\right\}$$

$$\text{Area } S_2 = \int_{x^2+y^2\leq 1} \sqrt{1+\|\nabla f\|^2}\ dxdy$$

$$\nabla f = (2x, 2y) \qquad \|\nabla f\|^2 = 4x^2+4y^2$$

$$= \int_{x^2+y^2\leq 1} \sqrt{1+4(x^2+y^2)}\ dxdy$$

$$\underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \int_{\substack{0\leq\rho\leq 1 \\ 0\leq\theta\leq 2\pi}} \sqrt{1+4\rho^2}\cdot\rho\ d\rho\, d\theta$$

$$\overset{RF}{=} 2\pi \int_0^1 (1+4\rho^2)^{1/2}\rho\ d\rho$$

$$\partial_\rho\left[\frac{(1+4\rho^2)^{3/2}}{12}\right] = \frac{3\cancel{4}}{\cancel{2}}(1+4\rho^2)^{1/2}\cdot\cancel{8}\rho$$

$$= \frac{2\pi}{12}\left[\frac{(1+4\rho^2)^{3/2}}{\cancel{12}}\right]_{\rho=0}^{\rho=1}$$

$$= \frac{\pi}{6}\left[5^{3/2}-1\right].$$

Q3. Outward flux of $\vec{F} = (x,y,z)$.

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\sigma \underset{\substack{\text{div}\\ \text{thm}}}{=} \int_{\Omega} \underset{\substack{\parallel \\ \partial_x x + \partial_y y + \partial_z z \,=\, 3}}{\text{div}\, \vec{F}}\; dx\, dy\, dz$$

$$= 3 \int_{\Omega} 1\, dx\, dy\, dz$$

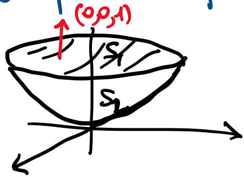
$$= 3 \int_{\overbrace{\underline{x^2+y^2} \leq z}\, \underbrace{\leq 1}} dx\, dy\, dz$$

$$= 3 \int_0^1 \left(\int_{x^2+y^2 \leq z} dx\, dy \right) dz$$

$$= 3 \int_0^1 \pi \cdot z \; dz$$

$$= 3\pi \left. \frac{z^2}{2} \right|_0^1 = \frac{3}{2}\pi.$$

Q4 Components of outward flux



$$\int_{S_1} \vec{F}\cdot\vec{n}\; d\sigma \; +$$

$$\int_{S_2} \vec{F}\cdot\vec{n}\; d\sigma$$

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\sigma$$

$$\int_{S_1} \vec{F}\cdot\vec{n}\, d\sigma = \int_{S_1} \overset{\overset{z}{\shortparallel}}{F_3}\, d\sigma$$

$$= \int_{S_1} 1\, d\sigma = \pi 1^2 = \pi. \quad ■$$

Do 5.8.5/7/8 $\quad x^2+6 \leq y^2+z^2 \leq 5x$

5.8.11 $\Omega = \{(x,y,z) : z^2+6 \leq x^2+y^2 \leq 5z\}$

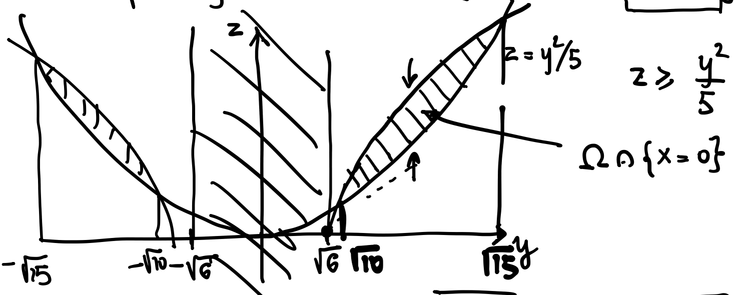
Q1 Is Ω a rotation solid? Draw Ω.

Q2. Compute outward flux of $\vec{F} = (x^2, y, z^2)$

(Q3. Compute components of Q2.)

Sol: Q1 Ω invariant by rots around z-axis.

$\Omega \cap$ plane yz $= \Omega \cap \{\underline{x=0}\} = \{z^2+6 \leq \underbrace{y^2 \leq 5z}\}$



$$z^2 \leq y^2 - 6 \iff -\sqrt{y^2-6} \leq z \leq \sqrt{y^2-6}$$

$$\boxed{z = \sqrt{y^2-6}}$$

$y^2 - 6 \geq 0 \quad y^2 \geq 6$

$y \geq \sqrt{6} \quad \lor \quad y \leq -\sqrt{6}$

$$\boxed{\sqrt{y^2-6} \geq \frac{y^2}{5}}$$

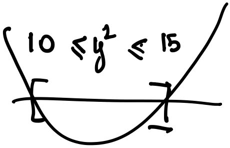
$$\Updownarrow$$

$$y^2 - 6 \geq \frac{y^4}{25} \Leftrightarrow y^4 - 25\overset{t}{y^2} + 150 \leq 0$$

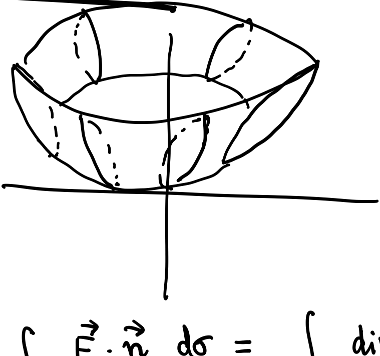
$\Delta = 625 - 600$

$= 25$

$$y^2_{1,2} = \frac{+25 \pm \overset{5}{\sqrt{25}}}{2} = 15, 10$$



$\sqrt{10} \leq y \leq \sqrt{15}$



Q2. $$\int_{\partial\Omega} \vec{F}\cdot\vec{n}\, d\sigma = \int_{\Omega} \operatorname{div}\vec{F}\, dx\, dy\, dz$$

$$\vec{F} = (x^2, y, z^2)$$

$$\operatorname{div}\vec{F} = \partial_x x^2 + \partial_y y + \partial_z z^2$$

$$= 2x + 1 + 2z$$

$$= \int_{z^2+6 \le \underline{x^2+y^2} \le 5z} (2x+2z+1)\, \underline{dx\,dy\,dz}$$

$$= \int_{\boxed{z^2+6 \le \rho^2 \le 5z},\ \underline{0 \le \theta \le 2\pi}} (2\rho\cos\theta + 2z+1)\rho\, d\rho\, d\theta\, dz$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$$\overset{RF}{=} \int_{z^2+6 \le \rho^2 \le 5z} \left(\int_0^{2\pi} (2\rho^2 \underline{\cos\theta} + \underline{2\rho z + \rho})\, d\theta \right)$$

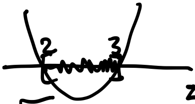
$$2\rho^2 \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} + (2\rho z + \rho)2\pi$$

$$= \int_{z^2+6 \le \rho^2 \le 5z} \boxed{2\pi}(2\rho z + \rho)\, d\rho\, dz$$

$$= 2\pi \underline{\int_2^3} \int_{\sqrt{z^2+6}}^{\sqrt{5z}} \rho(2z+1)\, d\rho \quad dz$$

$$\boxed{z^2 + 6 \le 5z} \Leftrightarrow \boxed{z^2 - 5z + 6 \le 0}$$

$\Delta = 25 - 24 = 1$



$$z_{1,2} = \frac{+5 \pm \sqrt{1}}{2}$$

$$= \cancel{2}\pi \int_2^3 (2z+1) \frac{\rho^2}{\cancel{2}} \Big|_{\sqrt{z^2+6}}^{\sqrt{5z}} dz$$

$$= \pi \int_2^3 (2z+1)\left(5z - (z^2+6)\right) dz = \dots$$