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Introduction to Diff. Eqns

What is a diff eqn?

eqn → identity with an unknown

diff → the unknown is a function $y = y(t)$ that appears in the eqn together with some of its derivatives:

$$y'(t), y''(t), \ldots \quad y^{(n)}(t)$$

where n is called **order of the eqn.**

Let's see some remarkable examples of models inducing a diff eqn.

Demographic Model

Goal: study the evol. (in time) of a population.

$P = P(t)$ = # of individuals at time t.

$P(t+h) - P(t)$ = var. of pop. in $[t, t+h]$

$$\frac{P(t+h)-P(t)}{h\,P(t)} = \text{growth rate} = r$$

$$\Big\downarrow$$

$$\frac{P'(t)}{P(t)} = r(t, P(t))$$

(first order diff eqn)

Two special cases are

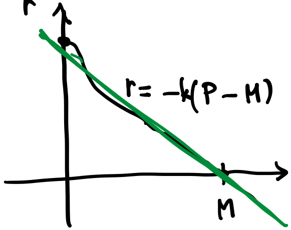
1. Malthus model: we assume $r(t,P) \equiv r_0$ constant

P verifies the eqn

$$\frac{P'(t)}{P(t)} = r_0 \iff \boxed{P'(t) = r_0 P(t)}$$

(example of first order linear eqn)

2. Logistic model: $r(t,P) = r(P) = k(M-P)$

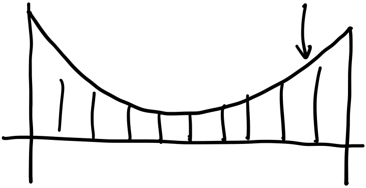


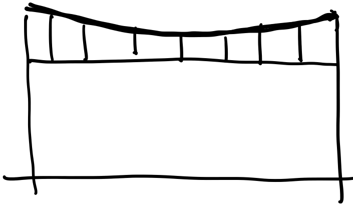
$r = -k(P-M) = k(M-P)$

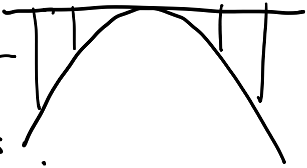
$$\Rightarrow \quad \frac{P'}{P} = k(M-P) \Leftrightarrow P' = kP(M-P)$$
$$= kMP - kP^2$$

(first order non linear eqn)

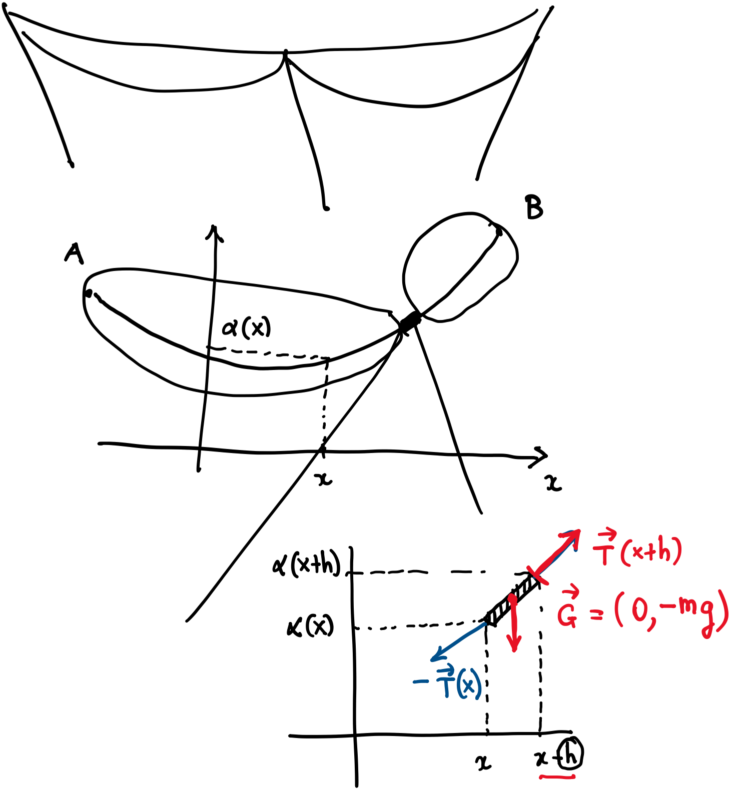
Catenary Pb:





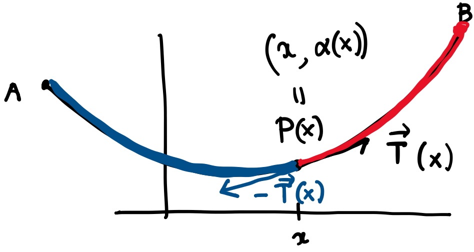


Pb: Det equilibrium pos of an heavy cable.



where

- m = mass of the little piece of cable
- g = 9.8 m/s^2

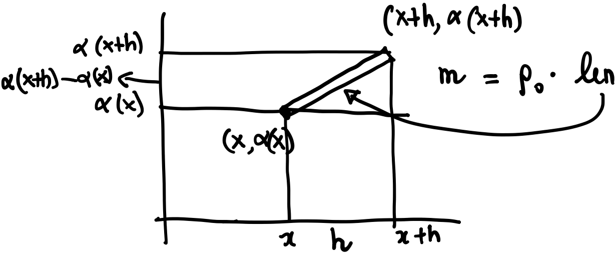


$\vec{T}(x)$ = tension exercised by the part of cable $\overset{\frown}{P(x)B}$

Equil. eqn :

$$\boxed{\vec{T}(x+h) - \vec{T}(x) + \vec{G} = \vec{0}}$$

Assumption: cable is homogeneous, ρ_0 is the mass density ($\rho_0 = \ldots \, kg/m$)



$m = \rho_0 \cdot$ lenght

$$\text{lenght} = \sqrt{h^2 + (\alpha(x+h) - \alpha(x))^2}$$

$$\overset{h>0}{=} h\sqrt{1 + \left(\frac{\alpha(x+h) - \alpha(x)}{h}\right)^2}$$

$$\simeq h\sqrt{1+\alpha'(x)^2}$$

$$\vec{G} = (0, -mg) = \Big(0, -\underbrace{\rho_0 h\sqrt{1+\alpha'(x)^2}\cdot g}_{m}\Big)$$

$$\vec{T}(x) = (\lambda(x), \mu(x))$$

$$\vec{T}(x+h) = (\lambda(x+h), \mu(x+h))$$

$$\vec{T}(x+h) - \vec{T}(x) + \vec{G} = \vec{0}$$

$$(\lambda(x+h), \mu(x+h)) - (\lambda(x), \mu(x)) + \left(0, -\rho_0 g h\sqrt{1+\alpha'^2}\right) = \vec{0}$$

$$\Leftrightarrow \begin{cases} \lambda(x+h) - \lambda(x) + 0 = 0 \\ \mu(x+h) - \mu(x) - \rho_0 g \boxed{h}\sqrt{1+\alpha'(x)^2} = 0 \end{cases}$$

$$\frac{\lambda(x+h) - \lambda(x)}{h} = 0 \quad \Rightarrow \lambda'(x) = 0$$

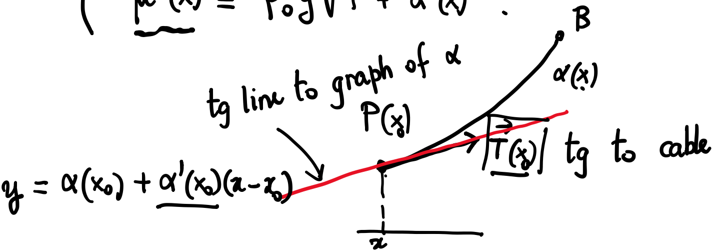
$$\Rightarrow \lambda(x) = \text{const} = \lambda_0$$

$$\underbrace{\frac{\mu(x+h) - \mu(x)}{h}}_{\mu'(x)} - \rho_0 g\sqrt{1+\alpha'(x)^2} = 0$$

$$\Rightarrow \mu'(x) - \rho_0 g\sqrt{1+\alpha'(x)^2} = 0$$

Conclusion: from equilibrium we get

$$\begin{cases}\lambda(x) = \lambda_0 \\ \underline{\mu'(x)} = \rho_0 g\sqrt{1+\alpha'(x)^2}\,.\end{cases} \qquad \vec{T}(x) = (\lambda(x), \mu(x))$$


B
$\alpha(x)$
tg line to graph of α
$P(x_0)$
$\vec{T}(x_0)$ tg to cable
$y = \alpha(x_0) + \underline{\alpha'(x_0)}(x - x_0)$
x_0

tg line to graph at x_0 // $\vec{T}(x_0) = (\lambda(x_0), \mu(x_0))$

$\Updownarrow$

they have the same ang coeff:

$$\alpha'(x_0) = \frac{\mu(x_0)}{\lambda(x_0)}$$

$$\Rightarrow \quad \alpha'(x) = \frac{\mu(x)}{\lambda(x) \equiv \lambda_0} = \frac{1}{\lambda_0}\mu(x)$$

$$\Rightarrow \quad \mu(x) = \lambda_0\,\alpha'(x)$$

$$\Rightarrow \quad \mu'(x) = \lambda_0\,\underline{\alpha''(x)}$$

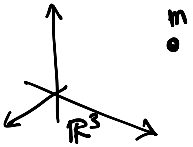
$$\Rightarrow \quad \boxed{\cancel{\lambda_0}\,\alpha''(x) = \frac{\rho_0 g}{\cancel{\lambda_0}}\sqrt{1+\alpha'(x)^2}}$$

If $v(x) = \alpha'(x)$, v verifies

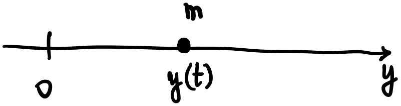
$$\boxed{v'(x) = \frac{\rho_0 g}{\lambda_0}\sqrt{1 + v(x)^2}}$$

(catenary eqn)

Newton Second Law



$$\boxed{m\vec{a} = \vec{F}}$$

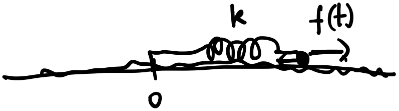


$a(t) = y''(t) \quad \Rightarrow \quad ma = my''(t)$

$F = F(t, y(t), y'(t))$

$ma = F \quad \iff \quad \boxed{my''(t) = F(t, y(t), y'(t))}$

(second order diff eqn.)



$$m y''(t) = -\underset{\overset{\vee}{0}}{k}\, y(t) - \gamma y'(t) + f(t)$$

$\blacksquare$

First Order Lin. Eqns

A First ord. lin eqn is an eqn of type

$$\boxed{y'(t) = a(t)\, y(t) + b(t)}$$

where $a(t), b(t) \in \mathcal{C}(I)$ are known (they are called coefficients). Usually we just write

$$y' = a(t)\, y + b(t)$$

Homogeneous eqn : $\boxed{y'(t) = a(t)\, y(t)}$

Example: $\underset{\overset{|||}{a(t)}}{y'(t) = r_0}\, y(t)$ (Malthus eqn

$\blacksquare$

Let y be a sol:

$$y'(t) = a(t)\, y(t)$$

$$\Updownarrow \quad y(t) \neq 0$$

$$\underline{\left(\log|y(t)|\right)'} = \boxed{\frac{y'(t)}{y(t)} = a(t)}$$

$$\Downarrow$$

$$\frac{1}{|y(t)|}\cdot\left(|y(t)|\right)' = \frac{1}{|y(t)|}\operatorname{sgn}(y(t))\,y'(t)$$

$$= \frac{\operatorname{sgn}(y(t))}{|y(t)|}\cdot y'(t)$$

$$\frac{\operatorname{sgn} x}{|x|} = \begin{cases} x>0 & \frac{1}{x} \\ x<0 & \frac{-1}{-x} = 1/x \end{cases}$$

$$= \frac{1}{y(t)}\cdot y'(t) = \frac{y'(t)}{y(t)}.$$

$$\Leftrightarrow \quad \left(\underline{\log|y(t)|}\right)' = a(t)$$

$$\Leftrightarrow \quad \boxed{\log|y(t)| = \int a(t)\,dt + c}$$

$c\in\mathbb{R}$

$$\Leftrightarrow \quad |y(t)| = e^{\int a(t)\,dt + c}$$

$$= e^{c}\, e^{\int a(t)\,dt}$$

$$|y(t)| = k\, e^{\int a(t)\,dt} \qquad k \in \mathbb{R},\ k > 0$$

$$\Leftrightarrow \quad y(t) = \pm k\, e^{\int a(t)\,dt}$$

$$\Leftrightarrow \quad \boxed{y(t) = K\, e^{\int a(t)\,dt} \qquad K \in \mathbb{R}\setminus\{0\}}$$

We notice that $y(t) \equiv 0$ is also a sol

$$y' = a(t)\,y \quad ?$$

$$\begin{array}{l} y' = a(t)\,y \\ \parallel \\ 0 = a(t)\cdot 0 = 0 \end{array}$$

Prop: y is a sol of

$$y' = a(t)\,y$$

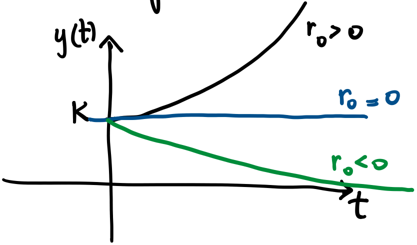
$$\Leftrightarrow \quad \boxed{y(t) = K\, e^{\int a(t)\,dt} \qquad K \in \mathbb{R}.}$$

(general sol / integral)

Ex: For Malthus eqn $y' = r_0 y$

$a(t) \equiv r_0$ (constant)

$$y(t) = K e^{\int r_0 \, dt} = \underset{y(0)=K}{\underset{\uparrow}{K}} e^{r_0 t}.$$



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Ex: Compute the gen sol of

$$y' = \overset{= a(t)}{\boxed{t^2}} y$$

Sol: The gen sol is

$$y(t) = K e^{\int t^2 \, dt} = K e^{t^3/3}, \quad K \in \mathbb{R}$$

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Non-hom. eqn: Let's now consider

$$y' = a(t) y + b(t).$$

To find all the sols we first do the following remark:

Prop: if $u = u(t)$ is a part sol of

$$y' = a(t) y + b(t) \qquad (*)$$

then, the gen sol of (*) is

$$y(t) = K\, e^{\int a(t)\,dt} + u(t), \quad K \in \mathbb{R}.$$

Proof: u solves (*)

$$\boxed{u' = a(t)u + b(t).} \quad (1) \; -$$

If y is any other sol of (*)

$$\boxed{y' = a(t)y + b(t)} \quad (2) \; =$$

$$\underbrace{(u-y)}_{v}' = u' - y' = a(t)\,(u-y)$$

$$v' = a(t)\, v \Rightarrow v \text{ solves the homog eqn}$$

$$\Rightarrow v = K e^{\int a(t)\,dt}$$

$$u - y = -K e^{\int a(t)\,dt} \qquad K \in \mathbb{R}$$

$$y = K e^{\int a(t)\,dt} + u(t) \quad \blacksquare$$

Pb: How can we compute a part sol $u = u(t)$?

Lagrange: $u(t) = K(t)\, e^{\int a(t)\,dt}$ looking for $K = K(t)$ s.t u solve

$$\boxed{u' = a(t)u + b(t)}$$

$$u = K e^{\boxed{\int a(t)\,dt \;=:\, A}} = K e^{A}$$

$$(K = K(t),\quad A = A(t)).$$

$$u' = \left(K e^{A}\right)' = K' e^{A} + K\left(e^{A}\right)'$$

$$= K' e^{A} + K e^{A} \cdot A'$$

$$\boxed{u' = K' e^{A} + K e^{A} \cdot a}$$

So u is a sol of $u' = au + b \iff$

$$\underbrace{K' e^{A} + a \cancel{K e^{A}}}_{u'} = a\, \underbrace{\cancel{K e}^{A}}_{u} + b$$

$$\iff \quad K' e^{A} = b$$

$$\iff \quad K' = e^{-A} b$$

$$\iff \quad K(t) = \int e^{-A(t)}\, b(t)\, dt$$

$$\boxed{K(t) = \int e^{-\int a(t)\,dt}\, b(t)\, dt}$$

Thm: The gen. sol of non-hom eqn

$$y' = a(t)y + b(t)$$

is
$$\boxed{y(t) = e^{\int a(t)dt}\left(K + \int e^{-\int a(t)dt}\, b(t)\, dt\right)}$$
$K \in \mathbb{R}$.

$$y(t) = K\underbrace{e^{\int a(t)dt}}_{} + \underset{\underset{\underbrace{K(t)}_{}\,\underbrace{e^{\int a(t)dt}}_{}}{\shortparallel}}{u(t)}$$

$$= \underbrace{e^{\int a(t)dt}\left(K + \int e^{-\int a(t)dt}\, b(t)\, dt\right)}_{}$$

Do Ex 6.5.1

Example 6.2.4. Find the gen int/sol of

$$y' - \frac{2}{t}y = 1 \quad , \quad t \in]0,+\infty[$$

Sol: We have a first ord. lin eqn

$$y' = \underset{a(t)}{\boxed{\frac{2}{t}}}\,y + \underset{b(t)}{\boxed{1}}$$

The gen int/sol is

$$y(t) = e^{\int a(t)\,dt}\left(K + \int e^{-\int a(t)\,dt}\,b(t)\,dt\right)$$

$$= e^{2\int \frac{\cancel{2}1}{t}\,dt}\left[K + \int e^{-\int \frac{2}{t}\,dt}\cdot 1\,dt\right]$$

$$= e^{2\log|t|}\left[K + \int e^{-2\log|t|}\,dt\right]$$

$$\overset{t>0}{=} e^{\cancel{2}\log t^{2}}\left[K + \int e^{-2\log t}\,dt\right]$$

$$= t^2\left(K + \underset{\frac{t^{-1}}{-1}}{\underset{\shortparallel}{\int t^{-2}\,dt}}\right)$$

$$= t^2\left(K - \frac{1}{t}\right) = t^2K - t$$

$K \in \mathbb{R}$ ∎

Ex.6.5.1

#3 $y' + 2ty = 2t^3$ $t \in \mathbb{R}$.

Sol: We have a 1st ord lin eqn

$$y' = \underbrace{\boxed{-2t}}_{a(t)}\,y + \underbrace{\boxed{2t^3}}_{b(t)}$$

The gen sol is

$$y(t) = e^{\int a(t)\,dt}\left(K + \int e^{\ominus\int a(t)\,dt}\cdot b(t)\,dt\right)$$

$$= e^{\int -2t\,dt}\left(K + \int e^{\int 2t\,dt}\cdot 2t^3\,dt\right)$$

$$= e^{-t^2}\left(K + \int e^{t^2}\cdot 2t^3\,dt\right)$$

$$\underset{\underset{(e^{t^2})'}{\shortparallel}}{(e^{t^2}\cdot 2t)}\ \underbrace{t^2}$$

$$= e^{-t^2}\left(K + e^{t^2}\cdot t^2 - \int \underset{\underset{(e^{t^2})'}{\shortparallel}}{e^{t^2}2t}\,dt\right)$$

$$= e^{-t^2}\left(K + e^{t^2}t^2 - e^{t^2}\right) \quad K\in\mathbb{R}$$

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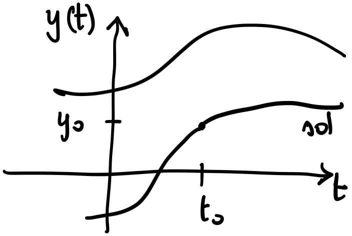
Rmks:

- In general, a diff eqn has ∞-many sols.
- A specific sol. can be identified by imposing some supplementary cond.

Cauchy Problem:

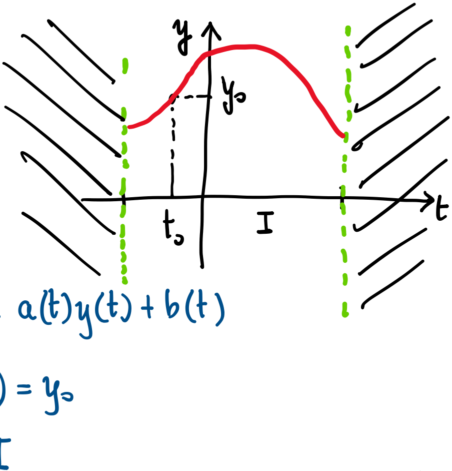
$$\begin{cases} y'(t) = a(t)y(t) + b(t) \\ y(t_0) = y_0 \end{cases}$$

with (t_0, y_0) known.



sol fulfilling the passage cond

Thm: Let, $a = a(t)$, $b = b(t) \in \mathcal{C}(I)$, I interval.



Then $\forall (t_0, y_0) \in I \times \mathbb{R}$ $\exists!$ sol of the CP

$$\begin{cases} y'(t) = a(t)y(t) + b(t) \\ y(t_0) = y_0 \end{cases}$$

defined $\forall t \in I$

Sol: The gen sol of eqn $y' = ay + b$ is

$$y = K e^{A(t)} + u(t) \qquad K \in \mathbb{R}$$

By imposing the poss. cond

$$y_0 = y(t_0) = K \underset{\substack{\vee \\ 0}}{e^{A(t_0)}} + u(t_0)$$

$$\iff K = \frac{y_0 - u(t_0)}{e^{A(t_0)}} \qquad \blacksquare$$

Ex 6.2.6

Solve

$$\begin{cases} y' - \dfrac{2y}{1-t^2} = t \\ y(2) = 0 \end{cases}$$

$t > 1$

$t \in \left]1, +\infty\right[$

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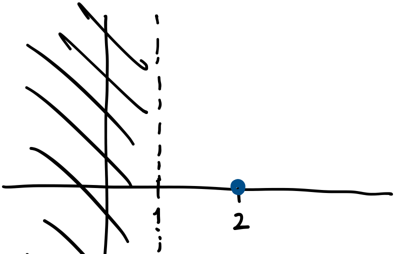
Sol: The eqn is a 1st order lin eqn

$$y' = \underbrace{\boxed{\frac{2}{1-t^2}}}_{a(t)} y + \underbrace{\boxed{t}}_{b(t)}$$

$a, b \in \mathcal{C}(\left]1, +\infty\right[)$

The gen int is

$y(t) =$



$$y(t) = e^{\int \frac{2}{1-t^2}\,dt}\left(K + \int e^{-\int \frac{2}{1-t^2}\,dt}\cdot t\,dt\right) \underset{(*)}{=}$$

$$\int \frac{2}{1-t^2}\,dt = \int \frac{2}{(1-t)(1+t)}\,dt$$

$$= \int \frac{1}{1-t} + \frac{1}{1+t}\Big)\,dt$$

$$= -\log|1-t| + \log|1+t|$$

$$= \log\left|\frac{t+1}{t-1}\right| \overset{t>1}{=} \log\frac{t+1}{t-1}$$

$$\overset{(*)}{=} e^{\log\frac{t+1}{t-1}}\left(K + \int e^{-\log\frac{t+1}{t-1}}\cdot t\,dt\right)$$

$$= \frac{t+1}{t-1}\left(K + \int \frac{t-1}{t+1}\,t\,dt\right) \overset{(**)}{=}$$

$$\int \frac{t-1}{t+1}(t+1-1)\,dt = \int t-1\,dt - \int \frac{\overset{+1-1}{t-1}}{t+1}\,dt$$

$$= \frac{t^2}{2} - t - \left(\int 1 - \frac{2}{t+1}\,dt\right)$$

$$= \frac{t^2}{2} - t - \left(t - 2\log(t+1)\right)$$

$$= \frac{t^2}{2} - 2t + 2\log(t+1)$$

$$y(t) \overset{(**)}{=} \frac{t+1}{t-1}\left(K + \frac{t^2}{2} - 2t + 2\log(t+1)\right) \quad K \in \mathbb{R}.$$

Now y is a sol of CP $\Leftrightarrow$ $y(2) = 0$

$$\frac{3}{1}\left(K + 2 - 4 + 2\log 3\right) = 0$$

$$K = 2 - 2\log 3. \quad \blacksquare$$

Do 6.5.1/2/3/4