Fundamentals of Mathematical Analysis 2 — MENG, ENSTP Yaounde

AY 2022/23 — Partial Exam (First Part on 40/100)

Exercise 1. Let

$$f(x, y) := (x^2 + y^2)^3 - x^4 + y^4, \ (x, y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x, y)$.
ii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find the eventual min/max points of f. What about $f(\mathbb{R}^2)$?

Exercise 2. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 + xy\right\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion.
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 3. Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y \leqslant 1 - |x|\}$ and $f(x, y) := 2xy - 2x^3 - y^3$.

i) Draw D and determine whether it is compact or not.
ii) Discuss the problem of searching for min/max of f on D.

Exercise 4. Let

$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1, \ y^2 + z = 1\}.$$

i) Show that $D \neq \varnothing$ is the zero set of a submersion (g_1, g_2).
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

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Exercise 5. Let
$$f(x,y) := x^4 + y^4 - 2(x-y)^2, \ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Find and classify the stationary points of f on $\mathbb{R}^2$.
iii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find the eventual min/max points of f. What about $f(\mathbb{R}^2)$?

Exercise 6. Let
$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2, \ y^2 + (z-2)^2 = 1\right\}.$$

i) Show that $D \neq \emptyset$ and it is the zeroes set of a submersion.
ii) Is D compact? Justify carefully.
iii) Find points of D (if any) at min/max distance to $\vec{0}$.

Exercise 7. Let
$$f(x,y,z) := (x^2+y^2+z^2)^2 - x^2 + y^2,\ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y,z)\to\infty_3} f(x,y,z)$.
ii) Find and classify the stationary points of f on $\mathbb{R}^3$.
iii) Discuss the problem to find min/max of f on $\mathbb{R}^3$. What about $f(\mathbb{R}^3)$?

Exercise 8. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ :\ x^2+y^2-z^2 = 1,\ x+y+2z = 0\}$.

i) Prove that $D \neq \emptyset$ and D is the set of zeroes of a submersion on D.
ii) Is D compact? Justify carefully your answer.
iii) Find points on D at minimum and at maximum distance to z axis.

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Exercise 9. Let

$$D := \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2x\}, \quad f(x,y) := x^3 + y^3 - 3xy.$$

i) Draw D. Is D open? Closed? Bounded? Compact? Connected? Justify your answer.
ii) Determine (if any) min/max of f on D. Determine $f(D)$, the image of D through f.

Exercise 10. Let

$$D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 - y^2 + 3z^2 = 2,\ x^2 + y^2 - z^2 - 4x = 0\right\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion.
ii) Is D compact? Justify your answer.
iii) Determine

$$\max_{(x,y,z)\in D} y.$$

Partial Exam (Second Part on 60/100)

Exercise 11. Solve the Cauchy problem

$$\begin{cases} y' = \dfrac{y^2 - 4}{t}, \\ y(1) = 0. \end{cases}$$

Exercise 12. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2)^{1/4} \leqslant z \leqslant 2 - x^2 - y^2\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.
iii) Compute the outward flux from D of the vector field $\vec{F} = (2x, 2y, 1)$, determining, in particular, the component of this flux on $D \cap \{z = 2 - x^2 - y^2\}$.

Exercise 13. Compute the area of the surface

$$S := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z^4 = x^2 + y^2, \ 0 \leqslant z \leqslant 1\right\}$$

PARTIAL EXAM (SECOND PART ON 60/100)

Exercise 14. Consider the equation

$$y' = \frac{e^y - 1}{t}, \ t \neq 0.$$

i) Determine the general integral.
ii) Determine the solution of the Cauchy problem $y(1) = -1$.

Exercise 15. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + 2y^2 \leqslant z \leqslant 4 - 3(x^2 + 2y^2)\}$.

i) Draw the set D into the space. Someone says: "D is a rotation volume with respect to the $z-$axis". Is it true or false?
ii) Compute the volume of D.
iii) Let $\vec{F} := (4xz, -y^2, yz)$. Compute the outgoing flux of $\vec{F}$ by D, determining in particular the components of the flow through the part of ∂D on the surfaces $z = x^2 + 2y^2$ and $z = 4 - 3(x^2 + 2y^2)$.

Exercise 16. Compute the area of the surface $S := \{(x, y, z) \in \mathbb{R}^3 \ : \ z = x^2 + 2y^2, 0 \leqslant z \leqslant 1\}$.

Partial Exam (Second Part on 60/100)

Exercise 17. Consider the second order equation
$$y'' - 2y' + y = e^{2t}.$$
i) Determine the general integral.
ii) Solve the Cauchy problem $y(0) = 1$, $y'(0) = 0$.
iii) For which $a \in \mathbb{R}$ there exists a solution such that $y(0) = 0$ and $y(1) = a$?

Exercise 18. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \leqslant 4,\ z \geqslant 1 - (x^2 + y^2),\ z \geqslant 0\}$.
i) Describe and draw D.
ii) Compute the volume of D
iii) Compute the outward flux by D of the vector field $\vec{F} = (x + y, y + z, z + x)$, determining, in particular, its component through $\partial D \cap \{x^2 + y^2 + z^2 = 4\}$.

Exercise 19. Compute the area of the surface $S := \{(x, y, z) \in \mathbb{R}^3 \ : \ x = 4 - (y^2 + z^2),\ 1 \leqslant x \leqslant 2\}$.

Partial Exam (Second Part on 60/100)

Exercise 20. Determine the general solution of the equation

$$y' = y - y^2.$$

Is it true or false that

$$\lim_{t \to +\infty} y(t) \in \mathbb{R}, \ \forall y \text{ solution?}$$

Justify your answer.

Exercise 21. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 - (x^2 + y^2) \leqslant z \leqslant \sqrt{1 - (x^2 + y^2)}\}$.

i) Draw D.
ii) Compute the volume of D.
iii) Let $\vec{F} = (x, y, z)$. Determine the flux of $\vec{F}$ through $D \cap \{z = 1 - (x^2 + y^2)\}$ with the normal pointing to the exterior of D.

Exercise 22. Compute the area of the surface $S := \{(x, y, z(\in \mathbb{R}^3 \ : \ z = \sqrt{x^2 + y^2}, \ a \leqslant z \leqslant b\}$, where $0 < a < b$ are fixed constants.

Solutions

Exercise 1. i) To compute the limit we write f in polar coordinates:

$$f(x,y) = (\rho^2)^3 - \rho^4(\cos\theta)^4 + \rho^4(\sin\theta)^4 \geqslant \rho^6 - \rho^4(\cos\theta)^4 \geqslant \rho^6 - \rho^4 \longrightarrow +\infty,$$

when $\rho = \|(x,y)\| \longrightarrow +\infty$. We conclude that $\exists \lim_{(x,y)\to\infty_2} f = +\infty$.

ii) By i) and since $f \in \mathscr{C}(\mathbb{R}^2)$, we have that there is no max for f on $\mathbb{R}^2$ while there is min f. Let $(x,y) \in \mathbb{R}^2$ be a min point. Since $(x,y) \in \mathbb{R}^2 = \mathrm{Int}(\mathbb{R}^2)$, according to Fermat theorem, $\nabla f(x,y) = \vec{0}$. Now,

$$\nabla f(x,y) = \vec{0}, \iff \begin{cases} 3(x^2+y^2)^2 2x - 4x^3 = 0, \\ \\ 3(x^2+y^2)^2 2y + 4y^3 = 0. \end{cases} \iff \begin{cases} x\left(3(x^2+y^2)^2 - 2x^2\right) = 0, \\ \\ y\left(3(x^2+y^2)^2 + 2y^2\right) = 0. \end{cases}$$

The second equation leads to the alternative $y = 0$ or $3(x^2+y^2)^2 + 2y^2 = 0$. In the first case, the first equation becomes

$$x(3x^4 - 2x^2) = 0, \iff x^3(3x^2-2) = 0, \iff x = 0, \vee x = \pm\sqrt{\frac{2}{3}}.$$

Thus we get stationary points $(0,0)$ and $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$. In the second case, we get $x = y = 0$, that is again point $(0,0)$.

Conclusion: min point for f is among $(0,0)$ and $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$. Since $f(0,0) = 0$ while $f\left(\pm\sqrt{\frac{2}{3}}, 0\right) = \frac{8}{27} - \frac{4}{9} = -\frac{4}{27} < 0$ we conclude that $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ are the global minimum points for f on $\mathbb{R}^2$.

Last question: since $\mathbb{R}^2$ is connected, $f(\mathbb{R}^2)$ is an interval, and because of i) and previous discussion on min for f we have $f(\mathbb{R}^2) = \left[-\frac{4}{27}, +\infty\right[$. $\square$

Exercise 2. i) For instance $(0,0,z) \in D$ iff $z^2 = 1$, thus $(0,0,\pm 1) \in D$ and $D \neq \emptyset$. D is also the zero set of $g(x,y,z) := x^2 + y^2 + z^2 - xy - 1$. This is a submersion on D iff

$$\nabla g \neq \vec{0}, \text{ on } D.$$

We have

$$\nabla g = \vec{0}, \iff \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z = 0, \end{cases} \iff (x,y,z) = (0,0,0) \notin D,$$

from which it follows that g is a submersion on D.

ii) Certainly, $D = \{g = 0\}$ is closed ($g \in \mathscr{C}$). Is it also bounded? We may see this by using spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi. \end{cases} \quad \rho^2 = x^2+y^2+z^2 = \|(x,y,z)\|^2.$$

Then, if $(x,y,z) \in D$ we have

$$\rho^2 = 1 + \rho^2\cos\theta\sin\theta(\sin\varphi)^2 = 1 + \frac{1}{2}\rho^2\sin(2\theta)(\sin\varphi)^2 \leqslant 1 + \frac{\rho^2}{2},$$

from which

$$\frac{\rho^2}{2} \leqslant 1, \iff \rho^2 = \|(x,y,z)\|^2 \leqslant 2.$$

Thus, D is bounded, hence compact.

iii) We have to minimize/maximize $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ or, which is equivalent (same min/max points), $f(x,y,z) = x^2+y^2+z^2$. According to i), we are in condition to apply Lagrange multipliers theorem. According to this result, at min/max points $(x,y,z) \in D$ we have

$$\nabla f = \lambda \nabla g, \iff \operatorname{rk} \begin{bmatrix} \nabla f(x,y,z) \\ \\ \nabla g(x,y,z) \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & 2z \\ \\ 2x-y & 2y-x & 2z \end{bmatrix} < 2.$$

This happens iff all 2×2 subdeterminats equal 0:

$$\begin{cases} 2x(2y-x) - 2y(2x-y) = 0, \\ \\ 2x2z - 2z(2x-y) = 0, \\ \\ 2y2z - 2z(2y-x) = 0, \end{cases} \iff \begin{cases} y^2 - x^2 = 0, \\ yz = 0, \\ xz = 0. \end{cases}$$

The first leads to $y = \pm x$, the second $y = 0$ (then $x = 0$) or $z = 0$. That is we have points $(0,0,z)$ and $(x,\pm x,0)$. Now

- $(0,0,z) \in D$ iff $z^2 = 1$, that is $(0,0,\pm 1)$.
- $(x,\pm x,0) \in D$ iff $2x^2 = 1 \pm x^2$. If $+$, $2x^2 = 1+x^2$, we get $x = \pm 1$, that is points $(1,1,0)$ and $(-1,-1,0)$. It $-$, $x^2 = \frac{1}{3}$, thus points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$.

Prom these we see that $(1,1,0)$ and $(-1,-1,0)$ are points at max distance to $\vec{0}$ while $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$ are points of D at min distance to $\vec{0}$. □

Exercise 3. i) D is a triangle with vertex $(-1,0)$, $(1,0)$ and $(0,1)$. In particular, it is closed (because defined by large inequalities) and bounded (we can also say: $0 \leqslant y \leqslant 1-|x| \leqslant 1$, and $1-|x| \geqslant 0$ implies $|x| \leqslant 1$). Therefore, it is compact.

ii) $f \in \mathscr{C}(D)$: according to Weierstrass' theorem, f admits both global min and max on D. Let's determine these points. If $(x,y) \in D$ is a min/max point then:

- either $(x,y) \in \operatorname{Int} D$, in this case (Fermat's theorem) $\nabla f(x,y) = \vec{0}$. Now,

  $$\nabla f(x,y) = \vec{0}, \iff \begin{cases} 2y - 6x^2 = 0, \\ \\ 2x - 3y^2 = 0, \end{cases} \iff \begin{cases} y = 3x^2, \\ \\ 2x - 27x^4 = 0, \iff x(2-27x^3) = 0, \end{cases}$$

  from which we obtain $x = 0$ or $x = \sqrt[3]{\frac{2}{27}} = \frac{\sqrt[3]{2}}{3}$. In the first case $y = 0$, in the second $y = \frac{\sqrt[3]{4}}{3}$. In conclusion, stationary points are $(0,0)$ and $\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4}}{3}\right)$. We notice that $(0,0) \notin \operatorname{Int} D$ while $\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4},}{3}\right) \in \operatorname{Int} D$.
- or $(x,y) \in D \backslash \operatorname{Int} D = A \cup B \cup C$ where

  $$A = \{(x,0) \ : \ -1 \leqslant x \leqslant 1\}, \quad B = \{(x,x+1) \ : \ -1 \leqslant x \leqslant 0\}, \quad C = \{(x,1-x) \ : \ 0 \leqslant x \leqslant 1\}.$$

  On A: $f(x,0) = -2x^3$ from which we easily deduce that $(1,0)$ is min point for f on A, $(-1,0)$ is max point for f on A.
  On B: $f(x,x+1) = 2x(x+1) - 2x^3 - (x+1)^3 = 2x^2+2x-2x^3-(x^3+3x^2+3x+1) = -3x^3-x^2-x-1 =: g(x)$. To determine min/max $g(x)$ for $x \in [-1,0]$ we compute $g'(x) = -9x^2-2x-1$ and discuss $g'(x) \geqslant 0$, iff $9x^2+2x+1 \leqslant 0$. Since $\Delta = 4-32 < 0$ we conclude that the previous inequality is never verified, thus $g' \leqslant 0$ on $[-1,0]$, that is $g \searrow$. We conclude that $(-1,0)$ is max for f on B and $(0,1)$ is min for f on B.
  On C: $f(x,1-x) = 2x(1-x) - 2x^3 - (1-x)^3 = 2x-2x^2-2x^3-(1-3x+3x^2-x^3) = -x^3-5x^2+5x-1 =: g(x)$. To determine min/max $g(x)$ for $x \in [0,1]$ we compute $g'(x) = -3x^2-10x+5$ and discuss $g'(x) \geqslant 0$, iff

$3x^2+10x-5 \leqslant 0$. Here $\Delta = 100+60 = 160 > 0$, thus solutions of the inequality are $\frac{-10-\sqrt{160}}{6} \leqslant x \leqslant \frac{-10+\sqrt{160}}{6} = \frac{-5+2\sqrt{10}}{3}$. Since $0 \leqslant x \leqslant 1$, we deduce that $g \nearrow$ for $0 \leqslant x \leqslant \frac{-5+2\sqrt{10}}{3}$. Thus $(0,1)$ and $(1,0)$ are min points, $\left(\frac{-5+2\sqrt{10}}{3}, \frac{8-2\sqrt{10}}{3}\right)$ is a max point.

Conclusion. Candidates for min are $\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4},}{3}\right)$ (from Int D), $(1,0)$ (from A), $(0,1)$ (from B) and again $(0,1)$, $(1,0)$ (from C). We have

$$f(1,0) = -2,\ f(0,1) = -1,\ f\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4},}{3}\right) = 2\frac{\sqrt[3]{8}}{27} - 2\frac{2}{27} - \frac{4}{27} = -\frac{4}{27}.$$

We conclude that $(0,1)$ is minimum point for f on D.

Candidates for max points are $\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4},}{3}\right)$ (from Int D), $(-1,0)$ (from A), the same from B, and $\left(\frac{-5+2\sqrt{10}}{3}, \frac{8-2\sqrt{10}}{3}\right)$ (from C). We have

$$f\left(\frac{\sqrt[3]{2}}{3}, \frac{\sqrt[3]{4},}{3}\right) = -\frac{4}{27},\ f(-1,0) = +2,\ f\left(\frac{-5+2\sqrt{10}}{3}, \frac{8-2\sqrt{10}}{3}\right) = \frac{2}{27}\left(-251+80\sqrt{10}\right) = 0.146\ldots$$

We conclude that $(-1,0)$ is max point for f on D. □

Exercise 4. i) $(x,y,0) \in D$ iff $x^2+y^2 = 1$ and y^2+1, that is $y = \pm 1$ hence $x^2 = 0$, that is $x = 0$. Thus $(0,\pm 1, 0) \in D$ and $D \neq \emptyset$. We may see $D = \{g_1 = 0,\ g_2 = 0\}$, where $g_1(x,y,z) = x^2+y^2-z^2-1$ and $g_2(x,y,z) = y^2+z-1$. Then, (g_1,g_2) is a submersion on D iff

$$2 = \mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & -2z \\ \\ 0 & 2y & 1 \end{bmatrix}.$$

Now, rank is < 2 iff all 2×2 sub matrices of previous matrix have null determinant, this leading to the system

$$\begin{cases} 4xy = 0, \\ 2x = 0, \\ 2y+4yz = 0, \end{cases} \iff \begin{cases} x = 0, \\ \\ y(1+2z) = 0, \end{cases} \iff (0,0,z),\ \vee\ (0,y,-\frac{1}{2}).$$

Now, $(0,0,z) \in D$ iff $-z^2 = 1$ and $z = 1$ (impossible!); $(0,y,-\frac{1}{2}) \in D$ iff $y^2 = \frac{5}{4}$ and $y^2 = \frac{3}{2}$ (impossible!). We conclude that (g_1,g_2) is a submersion on D.

ii) Clearly D is closed. Is it also bounded? Plugging $y^2 = 1-z$ into the first equation, we have

$$x^2 = z^2 - z + 1,$$

thus points

$$\left(\pm\sqrt{z^2-z+1}, \pm\sqrt{1-z}, z\right) \in D.$$

When $z \longrightarrow -\infty$ we see $\left(\pm\sqrt{z^2-z+1}, \pm\sqrt{1-z}, z\right) \longrightarrow \infty_3$. This tells that D is unbounded, hence not compact.

iii) We have to minimize/maximize $f = \sqrt{x^2+y^2+z^2}$ or, equivalently, $f = x^2+y^2+z^2$. By ii), D is unbounded, thus there is no max for f. However, since clearly $\lim_{(x,y,z)\to\infty_3} f = +\infty$, f has global minimum on D. Let $(x,y,z) \in D$ be a min point. By i), we can apply the Lagrange theorem: at (x,y,z) we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \mathrm{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} = 2,$$

iff the determinant of the last matrix equals 0,

$$-2y(-8xz) + 1(4xy - 4xy) = 0, \quad \Longleftrightarrow \quad xyz = 0, \quad \Longleftrightarrow \quad x = 0, \ \vee\ y = 0, \ \vee\ z = 0.$$

We get points $(0, y, z)$, $(x, 0, z)$, $(x, y, 0)$. Now:

$$(0, y, z) \in D, \quad \Longleftrightarrow \quad \begin{cases} y^2 - z^2 = 1, \\ \\ y^2 + z = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} z^2 + z = 0, \\ \\ y^2 = 1 - z \end{cases} \quad \Longleftrightarrow \quad (0, 0, \pm 1),\ (0, -1, \pm\sqrt{2}).$$

Similarly,

$$(x, 0, z) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 - z^2 = 1, \\ \\ z = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x^2 = 2, \\ \\ z = 1, \end{cases} \quad \Longleftrightarrow \quad (\pm\sqrt{2}, 0, 1).$$

Finally,

$$(x, y, 0) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 + y^2 = 1, \\ \\ y^2 = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x^2 = 0, \\ \\ y^2 = 1 \end{cases} \quad \Longleftrightarrow \quad (0, \pm 1, 0).$$

By computing distances we deduce that points of D at min distance to $\vec{0}$ are $(0, 0, \pm 1)$, $(0, \pm 1, 0)$. □

Exercise 5. i) We use polar coordinates. We have

$$f(x, y) = \rho^4 \left(\cos^4\theta + \sin^4\theta\right) - 2\rho^2 (\cos\theta - \sin\theta)^2 .$$

Now, $g(\theta) := \cos^4\theta + \sin^4\theta \geqslant 0$ for every $\theta \in [0, 2\pi]$. It is a continuous functions therefore it has a minimum. Let say that $C = g(\theta_{min})$ is the minimum value. We claim $C > 0$. Clearly $C \geqslant 0$. If $C = g(\theta_{min}) = \cos^4\theta_{min} + \sin^4\theta_{min} = 0$ then $\cos\theta_{min} = \sin\theta_{min} = 0$, but this is impossible. Thus $C > 0$. Then

$$f(x, y) \geqslant C\rho^4 - 2\rho^2(\cos\theta - \sin\theta)^2 \geqslant C\rho^4 - 8\rho^2 \longrightarrow +\infty,\ \rho \longrightarrow +\infty.$$

Since $\rho = \|(x, y)\|$ this means $\lim_{(x,y)\to\infty_2} f = +\infty$.

ii) We have

$$\partial_x f = 4x^3 - 4(x - y), \quad \partial_y f = 4y^3 + 4(x - y).$$

Clearly $\partial_x f, \partial_y f \in \mathscr{C}(\mathbb{R}^2)$, thus f is differentiable on $\mathbb{R}^2$. Stationary points fulfils,

$$\nabla f(x, y) = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 4x^3 - 4(x - y) = 0, \\ \\ 4y^3 + 4(x - y) = 0. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x - y = x^3, \\ \\ y^3 + x^3 = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y = -x, \\ \\ x^3 - 2x = 0, \end{cases}$$

From the second we get $x(x^2 - 2) = 0$, thus either $x = 0$ (then $y = 0$), or $x = \pm\sqrt{2}$ (and $y = \mp\sqrt{2}$). Thus, we get points $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

iii) By i) we know that f has a global minimum on $\mathbb{R}^2$. Since $\operatorname{Int}\mathbb{R}^2 = \mathbb{R}^2$, the global minimum must be a stationary point (Fermat's theorem). The possibilities are the stationary points $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$. Clearly $f(0, 0) = 0$ while

$$f(\pm\sqrt{2}, \mp\sqrt{2}) = 4 + 4 - 2(2\sqrt{2})^2 = 8 - 16 = -8.$$

We deduce that $(\pm\sqrt{2}, \mp\sqrt{2})$ are global minimum points for f.

By i) there is no max for f on $\mathbb{R}^2$. Finally, $f(\mathbb{R}^2) = [-8, +\infty[$. □

Exercise 6. i) For $z = 2$ we get

$$(x, y, 2) \in D, \iff \begin{cases} x^2 + y^2 = 4, \\ \\ y^2 = 1, \end{cases} \iff \begin{cases} y = \pm 1, \\ \\ x^2 = 3, \end{cases}$$

from which $(\pm\sqrt{3}, \pm 1, 2) \in D$ (four points). Clearly $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2 + y^2 - z^2$ and $g_2 = y^2 + (z-2)^2 - 1$. Clearly g_1, g_2 are differentiable (being polynomials). (g_1, g_2) is not a submersion where

$$\mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} < 2, \iff \mathrm{rk}\begin{bmatrix} 2x & 2y & -2z \\ \\ 0 & 2y & 2(z-2) \end{bmatrix} < 2, \iff \begin{cases} 4xy = 0, \\ 4x(z-2) = 0, \\ 4y(z-2) + 4yz = 0. \end{cases}$$

From first equation we get the alternative $x = 0$ or $y = 0$. In the first case, the second equation becomes $0 = 0$ (trivial) while the third one is $y(z-1) = 0$, that is $y = 0$ or $z - 1$. Thus we have points of type $(0, 0, z)$ and $(0, y, 1)$. Now,

$$(0, 0, z) \in D, \iff \begin{cases} z^2 = 0, \\ \\ (z-2)^2 = 1, \end{cases} \text{(impossible)}, \quad (0, y, 1) \in D, \iff \begin{cases} y^2 = 1, \\ \\ y^2 = 0, \end{cases} \text{(impossible)}.$$

In the second alternative $y = 0$. In this case, the third equation of initial system becomes trivial $0 = 0$ while the second leads to $x(z-2) = 0$, that is either $x = 0$ or $z = 2$. This gives points $(0, 0, z)$ and $(x, 0, 2)$. The first ones have already been checked and we know they are not on D. For the second ones,

$$(x, 0, 2) \in D, \iff \begin{cases} x^2 = 4, \\ \\ 0 = 1, \end{cases} \text{(impossible)}.$$

Conclusion: there are not points of D at which rank of $[\nabla g_1\ \nabla g_2]$ is less than 2, that is, (g_1, g_2) is a submersion on D.

ii) D is clearly closed being defined by equations involving continuous functions. Let's check that it is also bounded. From the second equation,

$$y^2 \leqslant 1,\ (z-2)^2 \leqslant 1, \implies 1 \leqslant z \leqslant 3, \implies z^2 \leqslant 9.$$

Then, from first equation,

$$x^2 = z^2 - y^2 \leqslant z^2 \leqslant 9,$$

thus

$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} \leqslant \sqrt{9 + 1 + 9},\ \forall (x, y, z) \in D.$$

This means that D is bounded, hence compact.

iii) We have to minimize/maximize $f(x, y, z) = \|(x, y, z)\|$ or, equivalently, $f(x, y, z) = \|(x, y, z)\|^2 = x^2+y^2+z^2$. Clearly, such an f is continuous and, by ii), D is compact. Thus f admits both min and max on D.

To determite min/max points, we apply Lagrange's theorem. By i), we can apply this theorem. Thus, if $(x, y, z) \in D$ is a min/max point for f, we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \mathrm{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3.$$

Since this last matrix is a 3×3 matrix, this condition boils down to

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 2(z-2) \end{bmatrix} = 2x\,(4y(z-2)+4yz) - 2x\,(4y(z-2)-4yz)$$

that is

$$xyz = 0.$$

This leads to the following alternatives: $x = 0$, that is points $(0, y, z)$, $y = 0$, that is points $(x, 0, z)$ and $z = 0$, that is points $(x, y, 0)$. Now,

$$(0, y, z) \in D, \iff \begin{cases} y^2 = z^2, \\ \\ y^2 + (z-2)^2 = 1, \end{cases} \iff \begin{cases} y^2 = z^2, \\ \\ 2z^2 - 2z + 3 = 0, \end{cases}$$

from which we see that there are not solutions. Again,

$$(x, 0, z) \in D, \iff \begin{cases} x^2 = z^2, \\ \\ (z-2)^2 = 1, \end{cases} \iff \begin{cases} z = 1, 3 \\ \\ x^2 = z^2, \end{cases}$$

from which we obtain points $(\pm 1, 0, 1)$, $(\pm 3, 0, 3)$. Finally,

$$(x, y, 0) \in D, \iff \begin{cases} x^2 + y^2 = 0, \\ \\ y^2 = -3, \end{cases}$$

for which there are not solutions.

Conclusion: since $f(\pm 1, 0, 1) = 2$ and $f(\pm 3, 0, 3) = 18$ we deduce that $(\pm 1, 0, 1)$ are min points while $(\pm 3, 0, 3)$ are max points. □

Exercise 7. i) In spherical coordinates,

$$f(x, y, z) = \rho^4 - \rho^2 (\cos\theta)^2 (\sin\varphi)^2 + \rho^2 (\sin\theta)^2 (\sin\varphi)^2 \geqslant \rho^4 - 2\rho^2 \longrightarrow +\infty,$$

when $\rho = \|(x, y, z)\| \longrightarrow +\infty$, and this means that $lim_{(x,y,z)\to\infty_3} f(x, y, z) = +\infty$.

ii) Point (x, y, z) is stationary point for f iff $\nabla f(x, y, z) = \vec{0}$, that is

$$\begin{cases} 2(x^2 + y^2 + z^2)2x - 2x = 0, \\ 2(x^2 + y^2 + z^2)2y + 2y = 0, \\ 2(x^2 + y^2 + z^2)2z = 0. \end{cases}$$

The third equation leads to the alternative $x^2 + y^2 + z^2 = 0$ (that is $(x, y, z) = (0, 0, 0)$) or $z = 0$. Plugging this into the other two equations we obtain

$$\begin{cases} z = 0, \\ x\left(2(x^2 + y^2) - 1\right) = 0, \\ y\left(2(x^2 + y^2) + 1\right) = 0, \end{cases} \iff \begin{cases} z = 0, \\ y = 0, \\ x(2x^2 - 1) = 0, \end{cases} \iff (x, y, z) = (0, 0, 0),\ \left(\pm\frac{1}{\sqrt{2}}, 0, 0\right).$$

iii) Since $f \in \mathscr{C}(\mathbb{R}^3)$ and by i), we conclude that there is no global max for f on $\mathbb{R}^3$ while there is minimum. Since f is also differentiable (clearly $\partial_x f, \partial_y f, \partial_z f \in \mathscr{C}(\mathbb{R}^3)$) and $\operatorname{Int} \mathbb{R}^3 = \mathbb{R}^3$, according to Fermat's theorem, any min point for f is a stationary point for f. By ii) we deduce that possible min points are $(0, 0, 0)$ (where $f = 0$) and $\left(\pm\frac{1}{\sqrt{2}}, 0, 0\right)$ (where $f = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$). We conclude that global min points for f on $\mathbb{R}^3$ are $\left(\pm\frac{1}{\sqrt{2}}, 0, 0\right)$.

Finally, $\mathbb{R}^3$ is connected, hence $f(\mathbb{R}^3)$ is an interval. By previous discussion $f(\mathbb{R}^3) = [-\frac{1}{4}, +\infty[$. □

Exercise 8. i) For instance $(x,y,0) \in D$ iff $x^2+y^2=1$ and $x+y=0$ that is $y=-x$ and $2x^2=1$, thus $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0\right) \in D$ and $D \neq \emptyset$. Now, $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2+y^2-z^2-1$ and $g_2 = x+y+2z$. We have that (g_1, g_2) is a submersion on D iff

$$\mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & -2z \\ \\ 1 & 1 & 2 \end{bmatrix} = 2,\ \forall (x,y,z) \in D.$$

Now, rank is <2 iff all 2×2 sub determinants of previous matrix are null, that is

$$\begin{cases} 2x-2y=0, \\ 4x+2z=0, \\ 4y+2z=0. \end{cases} \iff \begin{cases} y=x, \\ \\ z=-2x, \end{cases} \iff (x,y,z)=(x,x,-2x).$$

We have

$$(x,x,-2x) \in D,\ \ iff \begin{cases} x^2+x^2-4x^2=1, \\ \\ 2x-4x=0, \end{cases} \iff \begin{cases} x=0, \\ \\ 0=1, \end{cases}$$

which is manifestly impossible. We conclude that rank of $[\nabla g_1 \ \ \nabla g_2]^\perp$ is never <2 on D, thus (g_1,g_2) is a submersion on D.

ii) D is the set of zeroes of $g_1, g_2 \in \mathscr{C}(\mathbb{R}^3)$, thus it is closed. Is it bounded? We may notice that

$$(x,y,z) \in D, \iff \begin{cases} z = -\frac{x+y}{2}, \\ \\ x^2+y^2 = 1+z^2 = 1+\left(-\frac{x+y}{2}\right)^2 = 1+\frac{1}{4}(x^2+y^2+2xy). \end{cases}$$

From the secon equation,

$$\frac{3}{4}(x^2+y^2) = 1+\frac{1}{2}xy, \iff x^2+y^2 = \frac{4}{3}+\frac{2}{3}xy.$$

We claim that from this a bound for $\rho^2 := x^2+y^2$ follows. Indeed,

$$\rho^2 = \frac{4}{3}+\frac{2}{3}\rho^2\sin\theta\cos\theta = \frac{4}{3}+\frac{1}{3}\rho^2\sin(2\theta) \leqslant \frac{4}{3}+\frac{1}{3}\rho^2, \implies \frac{2}{3}\rho^2 \leqslant \frac{4}{3}, \implies \rho^2 \leqslant 2.$$

Therefore

$$x^2+y^2 \leqslant 2, \implies x^2 \leqslant 2,\ y^2 \leqslant 2, \implies |x|,|y| \leqslant \sqrt{2}.$$

From this

$$|z| = \left|-\frac{x+y}{2}\right| = \frac{1}{2}\left(|x|+|y|\right) \leqslant \frac{1}{2}\cdot 2\sqrt{2} = \sqrt{2}.$$

Finally,

$$\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} \leqslant \sqrt{2+2+2} = \sqrt{6},\ \forall (x,y,z) \in D.$$

This confirms that D is bounded, hence compact.

iii) We have to determine min/max of $f(x,y,z) = \sqrt{x^2+y^2}$ or, equivalently, $f(x,y,z) = x^2+y^2$. Since D is compact and f is continuous, f has both global min and max on D. Top determine these points we apply the Lagrange multipliers theorem. Since (g_1,g_2) is a submersion on D, at min/max points $(x,y,z) \in D$ for f we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \mathrm{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & 0 \\ 2x & 2y & -2z \\ 1 & 1 & 2 \end{bmatrix} = 2,$$

that is, iff

$$\det\begin{bmatrix} 2x & 2y & 0 \\ 2x & 2y & -2z \\ 1 & 1 & 2 \end{bmatrix} = 0, \iff 2x(4y+2z) - 2y(4x+2z) = 0, \iff z(x-y) = 0.$$

This produces points $z = 0$, that is $(x, y, 0)$ and points $y = x$, that is (x, x, z). Now:

$$(x, y, 0) \in D, \iff \begin{cases} x^2 + y^2 = 1, \\ \\ x + y = 0, \end{cases} \iff \begin{cases} y = -x, \\ \\ 2x^2 = 1, \end{cases} \iff \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0\right),$$

and

$$(x, x, z) \in D, \iff \begin{cases} 2x^2 - z^2 = 1, \\ \\ 2x + 2z = 0, \end{cases} \iff \begin{cases} z = -x, \\ \\ x^2 = 1, \end{cases} \iff (\pm 1, \pm 1, \mp 1).$$

Now, since

$$f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2} + \frac{1}{2} = 1, \quad f(\pm 1, \pm 1, \mp 1) = 1 + 1 = 2,$$

we deduce that $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0\right)$ are min points while $(\pm 1, \pm 1, \mp 1)$ are max points. $\square$

Exercise 9. i) Figure below. D is closed (defined by large inequalities involving continuous functions. Since $D\mathbb{R}^2$ and $D \neq \emptyset$ (both evident), D cannot be open. D is bounded: indeed, if $(x, y) \in D$ then $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 2x \leqslant 4$, thus $\|(x, y)\| = \sqrt{x^2 + y^2} \leqslant \sqrt{4 + 16} = \sqrt{20}$. Therefore, D is also compact (that is, closed and bounded). D is made of one single piece (clearly, any two points are joined by a line entirely contained in D).

ii) Clearly $f \in \mathscr{C}(D)$, D is compact, thus f admits both min and max on D. To determine these points, let $(x, y) \in D$ be a min/max point. We have the following alternative:

- either $(x, y) \in \operatorname{Int} D$: then, since $\partial_x f = 3x^2 - 3y \in \mathscr{C}(\mathbb{R}^2)$, $\partial_y f = 3y^2 - 3x \in \mathscr{C}(\mathbb{R}^2)$, f is differentiable, thus, by Fermat's theorem, $\nabla f(x, y) = \vec{0}$. Now,

$$\nabla f(x, y) = \vec{0}, \iff \begin{cases} 3x^2 - 3y = 0, \\ \\ 3y^2 - 3x = 0, \end{cases} \iff \begin{cases} y = x^2, \\ \\ x^4 - x = 0. \end{cases}$$

  The second equation leads to $x = 0$ or $x^3 = 1$, that is $x = 1$. Thus we get points $(0, 0)$ and $(1, 1)$. Since $(0, 0) \notin \operatorname{Int} D$ we discard this point, while $(1, 1) \in \operatorname{Int} D$.

- or $(x, y) \in D \backslash \operatorname{Int} D = A \cup B \cup C$ where

$$A = \{(x, 0) \ : \ 0 \leqslant x \leqslant 2\}, \quad B = \{(x, 2x) \ : \ 0 \leqslant x \leqslant 2\}, \quad C = \{(2, y) \ : \ 0 \leqslant y \leqslant 4\}.$$

  On A,

$$f(x, 0) = x^3,$$

  which is minimum for $x = 0$ and maximum for $x = 2$. Thus, candidate min is $(0, 0)$ while candidate max is $(2, 0)$.

  On B,

$$f(x, 2x) = x^3 + 8x^3 - 6x^2 = 9x^3 - 6x^2 =: g(x).$$

  We have $g'(x) = 27x^2 - 12x = 3x(9x - 4) \geqslant 0$ iff (for $0 \leqslant x \leqslant 2$), $x \geqslant \frac{4}{9}$. Thus $x = 0, 2$ are max points, $x = \frac{4}{9}$ is min point. For f this means that points $(0, 0)$ and $(2, 4)$ are possible max points while $(\frac{4}{9}, \frac{8}{9})$ is a possible min point.

On C we have $f(2,y) = 8 + y^3 - 6y =: g(y)$. We have $g'(y) = 3y^2 - 6 = 3(y^2 - 2) \geqslant 0$ iff $y^2 \geqslant 2$ that is, for $0 \leqslant y \leqslant 4$, when $\sqrt{2} \leqslant y \leqslant 4$. Thus, $y = 0, 4$ are max points, while $y = \sqrt{2}$ is min point for f on C. We conclude that $(2, \sqrt{2})$ is possible min point for f while $(2, 0)$ and $(2, 4)$ are possible max points for f.

We can now draw the conclusion. Possible max points are $(1, 1)$ (from Ind D), $(2, 0)$ (from A), $(0, 0)$, and $(2, 4)$ (from B), $(2, 0)$ and $(2, 4)$ (from C). Since

$$f(1,1) = -1, \quad f(2,0) = 8, \quad f(0,0) = 0, \quad f(2,4) = 48,$$

we conclude that $(2, 4)$ is global max for f on D.

Possible min points are $(1, 1)$ (from Ind D), $(0, 0)$ (from A), $(\frac{4}{9}, \frac{8}{9})$ (from B), $(2, \sqrt{2})$ (from C). Since

$$f(1,1) = -1, \quad f(0,0) = 0, \quad f\left(\frac{4}{9}, \frac{8}{9}\right) = \frac{4^3 + 8^3 - 3^3 \cdot 4 \cdot 8}{9^3} = -\frac{288}{729}, \quad f(2, \sqrt{2}) = 8 + 2\sqrt{2} - 3 \cdot 2\sqrt{2} > 0,$$

from which we deduce that $(1, 1)$ is the global min point. □

Exercise 10. i) For example $(0, y, z) \in D$ iff

$$\begin{cases} -y^2 + 3z^2 = 2, \\ y^2 - z^2 = 0, \end{cases} \iff \begin{cases} z^2 = y^2, \\ z^2 = 1, \end{cases} \iff \begin{cases} z = \pm 1, \\ y = \pm 1, \end{cases}$$

that is $(0, \pm 1, \pm 1) \in D$ (four points, all possible combinations of sign $\pm$). Clearly $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2 - y^2 + 3z^2 - 2$ and $g_2 = x^2 + y^2 - z^2 - 4x$. Now, (g_1, g_2) is a submersion iff

$$\operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & -2y & 6z \\ 2(x-2) & 2y & -2z \end{bmatrix} = 2, \ \forall (x, y, z) \in D.$$

This is false iff all 2×2 subdeterminants vanish, that is, iff

$$\begin{cases} 4xy + 4y(x-2) = 0, \\ -4xz - 12z(x-2) = 0, \\ 4yz - 12yz = 0, \end{cases} \iff \begin{cases} y(x-1) = 0, \\ z(2x-1) = 0, \\ yz = 0. \end{cases}$$

The last equation leads to the alternative $y = 0$ or $z = 0$. In the first case the system reduces to

$$\begin{cases} y = 0, \\ z(2x-1) = 0, \end{cases} \iff \begin{cases} y = 0, \\ z = 0, \end{cases} \vee \begin{cases} y = 0, \\ x = \frac{1}{2}. \end{cases}$$

Therefore, we have points $(x, 0, 0)$ and $(\frac{1}{2}, 0, z)$. Now

$$(x, 0, 0) \in D, \iff \begin{cases} x^2 = 2, \\ x^2 - 4x = 0, \end{cases} \quad \text{(impossible)},$$

and

$$\left(\frac{1}{2}, 0, z\right) \in D, \iff \begin{cases} \frac{1}{4} + 3z^2 = 2, \\ \frac{1}{4} - z^2 - 2 = 0, \end{cases} \quad \text{(impossible)}.$$

In the second alternative, $z = 0$, the system reduces to

$$\begin{cases} z = 0, \\ y(x-1) = 0, \end{cases} \iff \begin{cases} z = 0, \\ y = 0 \end{cases} \vee \begin{cases} z = 0, \\ x = 1. \end{cases}$$

Solutions are points $(x, 0, 0)$ and $(1, y, 0)$. Now,

$$(x,0,0) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 = 2, \\ \\ x^2 - 4x = 0, \end{cases} \quad \text{(impossible)},$$

and

$$(1,y,0) \in D, \quad \Longleftrightarrow \quad \begin{cases} 1 - y^2 = 2, \\ \\ 1 + y^2 - 4 = 0, \end{cases} \quad \text{(impossible)}.$$

Conclusion: since there are no points on D such that $\mathrm{rk}[\nabla g_1 \ \nabla g_2]^\perp < 2$, we conclude that (g_1, g_2) is a submersion on D.

ii) Clearly D is closed being defined by equations involving continuous functions. To see whether D is compact or less, we need to check if D is bounded. Notice that,

$$(x,y,z) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 - y^2 + 3z^2 = 2, \\ \\ x^2 + y^2 - z^2 - 4x = 0, \end{cases} \quad \Longrightarrow \quad 2x^2 + 2z^2 - 4x = 2, \ \ (x-1)^2 + z^2 = 3$$

from which $|x - 1| \leqslant \sqrt{3}$, and $z^2 \leqslant 3$. The first says $1 - \sqrt{3} \leqslant x \leqslant 1 + \sqrt{3}$, thus $x^2 \leqslant 9$. Plugging this into one of the two equations for D we have

$$y^2 = x^2 + 3z^2 - 2 \leqslant 9 + 9 - 2 = 16,$$

thus $\|(x, y, z)\| \leqslant \sqrt{9 + 3 + 16} = \sqrt{28}$ for every $(x, y, z) \in D$. This means that D is bounded, hence it is compact.

iii) We have to determine min/max of $f(x, y, z) = y$. Clearly, $f \in \mathscr{C}(D)$ and D is compact, thus f has both min and max on D. By i), to determine these points we can apply the Lagrange theorem. If $(x, y, z) \in D$ is any min/max point for f then

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk} \begin{bmatrix} 0 & 1 & 0 \\ 2x & -2y & 6z \\ 2(x-2) & 2y & -2z \end{bmatrix} < 3,$$

and this is equialent to

$$0 = \det \begin{bmatrix} 0 & 1 & 0 \\ 2x & -2y & 6z \\ 2(x-2) & 2y & -2z \end{bmatrix} = -1(-4xz - 12(x-2)z) = 8z(2x-3), \quad \Longleftrightarrow \quad z = 0, \ \vee \ x = \frac{3}{2}.$$

This leads to points $(x, y, 0)$ and $\left(\frac{3}{2}, y, z\right)$. Let's check when they belong to D. We have

$$(x,y,0) \in D, \quad \Longleftrightarrow \quad \begin{cases} x^2 - y^2 = 2, \\ \\ x^2 + y^2 - 4x = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y^2 = x^2 - 2, \\ \\ x^2 - 2x - 1 = 0, \end{cases}$$

We get $x = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$. For $x = 1 + \sqrt{2}$, we have $y^2 = x^2 - 2 = 1 + 2 + 2\sqrt{2} - 2 = 1 + 2\sqrt{2}$, thus $y = \pm\sqrt{1 + 2\sqrt{2}}$, or points $\left(1 + 2\sqrt{2}, \pm\sqrt{1 + 2\sqrt{2}}, 0\right)$. For $x = 1 - \sqrt{2}$ we have $y^2 = x^2 - 2 = 1 + 2 - 2\sqrt{2} - 2 = 1 - 2\sqrt{2} < 0$, no solutions. Now,

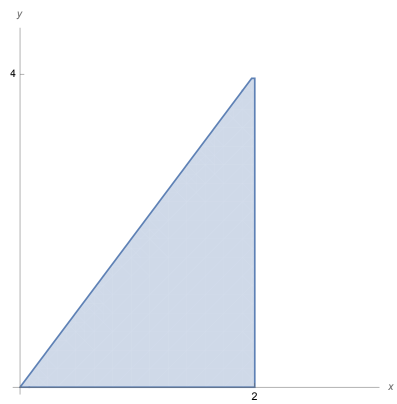
$$\left(\frac{3}{2}, y, z\right) \in D, \quad \Longleftrightarrow \quad \begin{cases} \frac{9}{4} - y^2 + 3z^2 = 2, \\ \\ \frac{9}{4} + y^2 - z^2 - 6 = 0. \end{cases}$$

Summing the two equations we get

$$\frac{9}{2} + 2z^2 = 2, \quad \Longleftrightarrow \quad 2z^2 = 2 - \frac{9}{2} < 0,$$

thus no solutions. We conclude that the possible min/max points are $\left(1 + 2\sqrt{2}, \pm\sqrt{1 + 2\sqrt{2}}, 0\right)$. Clearly, + gives max point while − is the min point. □

Exercise 9. i) D is defined by large inequalities, therefore it is closed. Clearly $D \neq \emptyset, \mathbb{R}^2$, thus D is not open. It is bounded, because $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 4$. Hence, D is compact. Clearly, D is also connected.



ii) Since f is continuous, D is compact, f has both global min and max on D. Let $(x, y) \in D$ be a min/max point. Then

- either $(x, y) \in \operatorname{Int} D$, then, according to Fermat's theorem, $\nabla f(x, y) = \vec{0}$. Now,

$$\nabla f = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 3x^2 - 3y = 0, \\ \\ 3y^2 - 3x = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y = x^2, \\ \\ x^4 - x = 0, \end{cases}$$

  Since $x^4 - x = 0$ iff $x(x^3 - 1) = 0$, that is $x = 0$ or $x = 1$, we get points $(0, 0)$ and $(1, 1)$. The former does not belong to $\operatorname{Int} D$ while the latter belongs to $\operatorname{Int} D$.
- or $(x, y) \in D \setminus \operatorname{Int} D = A \cup B \cup C$ where

$$A := \{(x, 0) \ : \ 0 \leqslant x \leqslant 2\}, \quad B := \{(2, y) \ : \ 0 \leqslant y \leqslant 4\}, \quad C := \{(x, 2x) \ : \ 0 \leqslant x \leqslant 2\}.$$

  On A: $f(x, 0) = x^3$, that takes its min value for $x = 0$ and max value for $x = 2$, thus $(0, 0)$ is the min point for f on A and $(2, 0)$ is the max point for f on A.

  On B: $f(2, y) = 8 + y^3 - 6y =: g(y)$, $y \in [0, 4]$. We have $g'(y) = 3y^2 - 6 \geqslant 0$, iff $y^2 \geqslant 2$, that is $(y \in [0, 4])$, $y \geqslant \sqrt{2}$. Thus $g \searrow$ on $[0, \sqrt{2}]$ and $g \nearrow$ on $[\sqrt{2}, 4]$. We conclude that $y = 0, 4$ are max points for g on $[0, 4]$ while $x = \sqrt{2}$ is min point for g on $[0, 4]$. This means that $(2, 0), (2, 4)$ are max points for f on B and $(2, \sqrt{2})$ is min point for f on B.

  On C: $f(x, 2x) = x^3 + 8x^3 - 6x^2 = 9x^3 - 6x^2 =: g(x)$, $x \in [0, 2]$. We have $g'(x) = 27x^2 - 12x = x(27x - 12) \geqslant 0$ iff $x \leqslant 0$ or $x \geqslant \frac{12}{27} = \frac{4}{9}$. Therefore, on $[0, 2]$, we have $g \searrow$ on $[0, \frac{4}{9}]$ and $g \nearrow$ on $[\frac{4}{9}, 2]$. We conclude that $x = 0, 2$ are max points for g on $[0, 2]$ and $x = \frac{4}{9}$ is min point for g on $[0, 2]$, that is $(0, 0), (2, 4)$ are max points for f on C and $(\frac{4}{9}, \frac{8}{9})$ is min point for f on C.

Conclusion. Possible candidates for min are points $(1,1)$, $(0,0)$, $(2,\sqrt{2})$, $\left(\frac{4}{9},\frac{8}{9}\right)$. We have

$$f(1,1)=-1,\quad f(0,0)=0,\quad f(2,\sqrt{2})=8+\sqrt{8}-6\sqrt{2},\quad f\left(\frac{4}{9},\frac{8}{9}\right)=\frac{288}{729}.$$

From this we deduce that $(1,1)$ is the global min point for f on D.

Possible candidates for max are points $(1,1)$, $(2,0)$, $(2,4)$, $(0,0)$. We have

$$f(1,1)=-1,\quad f(2,0)=8,\quad f(2,4)=48,\quad f(0,0)=0,$$

from which we see that $(2,4)$ is the global max point for f on D.

Finally, $f(D)=[-1,48]$. □

Exercise 10. i) If $z=0$, that is taking a point of type $(x,y,0)$, we have $(x,y,0)\in D$ iff

$$\begin{cases} x^2-y^2=2, \\ \\ x^2+y^2-4x=0, \end{cases}$$

from which $2x^2-4x=2$, or $x^2-2x=1$, leading to $x=1\pm\sqrt{2}$. Plugging this into the first equation we have $y^2=x^2-2=1+2\pm 2\sqrt{2}-2=1\pm 2\sqrt{2}$. For $-$ there are no solutions, while for $+$ we have $y^2=1+2\sqrt{2}$ that is $y=\pm\sqrt{1+2\sqrt{2}}$. In conclusion, $(1+\sqrt{2},\pm\sqrt{1+2\sqrt{2}},0)\in D$ and $D\neq\emptyset$.

We notice that $D=\{g_1=0,\ g_2=0\}$ where $g_1=x^2-y^2+3z^2-2$ and $g_2=x^2+y^2-z^2-4x$. We recall that (g_1,g_2) is a submersion at (x,y,z) iff

$$2=\mathrm{rk}\begin{bmatrix} \nabla g_1(x,y,z) \\ \\ \nabla g_2(x,y,z) \end{bmatrix}=\mathrm{rk}\begin{bmatrix} 2x & -2y & 6z \\ \\ 2x-4 & 2y & -2z \end{bmatrix}.$$

Now, this fails iff all 2×2 sub determinants vanish, that is, iff

$$\begin{cases} 4xy+4y(x-2)=0, \\ -4xz-4z(x-2)=0, \\ 4yz-12yz=0 \end{cases} \iff \begin{cases} y(x-1)=0, \\ z(x-1)=0, \\ yz=0. \end{cases}$$

The first equation leads to the alternative $y=0$ or $x=1$. In the first case we get the subsystem

$$\begin{cases} y=0, \\ z(x-1)=0, \\ 0=0 \end{cases} \iff (x,0,0),\quad (1,0,z).$$

In the second case, we obtain the subsystem

$$\begin{cases} x=1, \\ 0=0, \\ yz=0, \end{cases} \iff (1,0,z),\quad (1,y,0).$$

We check how many of these points belong to D. We have

$$(x,0,0)\in D,\quad\iff\quad \begin{cases} x^2=2, \\ \\ x^2-4x=0, \end{cases}\quad \text{impossible.}$$

Next,

$$(1,0,z)\in D,\quad\iff\quad \begin{cases} 3z^2=1, \\ \\ -z^2=3 \end{cases}\quad \text{impossible.}$$

Finally,

$$(1, y, 0) \in D, \iff \begin{cases} 1 - y^2 = 2, \\ \\ 1 + y^2 - 4 = 0, \end{cases} \quad \text{impossible.}$$

We conclude that (g_1, g_2) is a submersion on D.

ii) D is definitely closed (being defined by equations involving continuous functions). Is it also bounded? If $(x, y, z) \in D$, then summing the two equations we get

$$2x^2 + 2z^2 - 4x = 2, \iff x^2 - 2x + z^2 = 1, \iff (x-1)^2 + z^2 = 2,$$

thus

$$|x - 1| \leqslant \sqrt{2}, \quad |z| \leqslant \sqrt{2}.$$

In particular, $1 - \sqrt{2} \leqslant x \leqslant 1 + \sqrt{2}$, from which we can say $-3 \leqslant x \leqslant 3$. Then

$$y^2 = -x^2 + z^2 + 4x \leqslant z^2 + 4x \leqslant 2 + 12 = 14,$$

thus

$$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} \leqslant \sqrt{9 + 14 + 2}, \ \forall (x, y, z) \in D.$$

This means that D is bounded, hence compact.

iii) Let $f(x, y, z) := y$. Clearly, $f \in \mathscr{C}(D)$, thus, according to Weierstrass' thm, f has both global min and max on D. To determine these points, thanks to i), we apply Lagrange's multipliers theorem. At $(x, y, z) \in D$ min/max for f we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2,$$

that is, iff

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 & 0 \\ 2x & -2y & 6z \\ 2x - 4 & 2y & -2z \end{bmatrix} = -(-4xz - 12z(x-2)) = 4z(4x - 6).$$

This leads to $z = 0$ (thus, points $(x, y, 0)$ or $x = \frac{3}{2}$ (thus, points $(\frac{3}{2}, y, z)$). Now, by what seen in i),

$$(x, y, 0) \in D, \iff (1 + \sqrt{2}, \pm\sqrt{1 + \sqrt{2}}, 0).$$

Moreover

$$\left(\frac{3}{2}, y, z\right) \in D, \iff \begin{cases} \frac{9}{4} - y^2 + 3z^2 = 2, \\ \\ \frac{9}{4} + y^2 - z^2 = 6, \end{cases} \iff \begin{cases} z^2 = \frac{7}{4}, \\ \\ y^2 = \frac{11}{2}, \end{cases} \iff \left(\frac{3}{2}, \pm\sqrt{\frac{11}{2}}, \pm\sqrt{\frac{7}{4}}\right)$$

(four points, all possible combinations of sign). Now, since

$$f(1 + \sqrt{2}, \pm\sqrt{1 + \sqrt{2}}, 0) = \pm\sqrt{1 + \sqrt{2}}, \quad f\left(\frac{3}{2}, \pm\sqrt{\frac{11}{2}}, \pm\sqrt{\frac{7}{4}}\right) = \pm\sqrt{\frac{11}{2}}$$

it is clear that min and max are, respectively, points $\left(\frac{3}{2}, -\sqrt{\frac{11}{2}}, \pm\sqrt{\frac{7}{4}}\right)$ (min points), $\left(\frac{3}{2}, +\sqrt{\frac{11}{2}}, \pm\sqrt{\frac{7}{4}}\right)$ (max points). □

Exercise 11. The equation is a separable variables equation,

$$y' = \frac{1}{t}(y^2 - 4) = a(t) f(y),$$

where $f(y) = y^2 - 4$. Constant solutions are $y \equiv C$ such that $0 = \frac{1}{t}(C^2 - 4)$, that is $C^2 - 4 = 0$, or $C = \pm 2$. Since the solution of the proposed CP is $y(1) = 0$, the solution y cannot be constant (otherwise $y \equiv \pm 2$ but $y(1) = 0 \neq \pm 2$). Thus, the solution have to be found by separation of variables:

$$y' = \frac{1}{t}(y^2 - 4), \quad \Longleftrightarrow \quad \frac{y'}{y^2 - 4} = \frac{1}{t}, \quad \Longleftrightarrow \quad G(y)' = \frac{1}{t},$$

where

$$G(y) = \int \frac{1}{y^2 - 4}\, dy = \frac{1}{4} \int \frac{1}{y-2} - \frac{1}{y+2}\, dy = \frac{1}{4}\left(\log|y-2| - \log|y+2|\right) = \frac{1}{4}\log\left|\frac{y-2}{y+2}\right|.$$

Therefore

$$\left(\frac{1}{4}\log\left|\frac{y-2}{y+2}\right|\right)' = \frac{1}{t}, \quad \Longleftrightarrow \quad \frac{1}{4}\log\left|\frac{y-2}{y+2}\right| = \int \frac{1}{t}\, dt + c = \log|t| + c.$$

Imposing the passage condition $y(1) = 0$ we easily get $c = 0$, thus

$$\frac{1}{4}\log\left|\frac{y-2}{y+2}\right| = \log|t|, \quad \Longleftrightarrow \quad \left|\frac{y-2}{y+2}\right| = t^4, \quad \Longleftrightarrow \quad \frac{y-2}{y+2} = \pm t^4.$$

Again, by $y(1) = 0$ we get $-1 = \pm 1$, thus sign is $-$ and

$$\frac{y-2}{y+2} = -t^4, \quad \Longleftrightarrow \quad y - 2 = -t^4(y+2), \quad \Longleftrightarrow \quad y(1+t^4) = 2(1-t^4), \quad \Longleftrightarrow \quad y = 2\frac{1-t^4}{1+t^4},$$

which is the sought solution. □

Exercise 12. i) Notice that $D \cap \{x = 0\} = \{(0, y, z) \ : \ \sqrt{|y|} \leqslant z \leqslant 2 - y^2\}$.

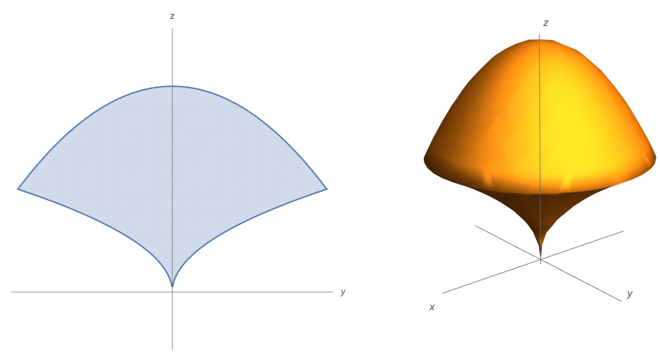


FIGURE 1. $D \cap \{x = 0\}$ (left) and D (right)

ii) We have

$$\begin{aligned}
\lambda_3(D) &= \int_D 1\, dxdydz = \int_{(x^2+y^2)^{1/4} \leqslant z \leqslant 2 - x^2 - y^2} 1\, dxdydz \\
&\overset{RF}{=} \int_{(x^2+y^2)^{1/4} \leqslant 2 - (x^2+y^2)} \left(2 - (x^2+y^2) - (x^2+y^2)^{1/4}\right)\, dxdy \\
&\overset{pol.coords}{=} \int_{\sqrt{\rho} \leqslant 2 - \rho^2} \left(2 - \rho^2 - \sqrt{\rho}\right) \rho\, d\rho d\theta \\
&\overset{RF}{=} 2\pi \int_{\sqrt{\rho} \leqslant 2 - \rho^2} \left(2\rho - \rho^3 - \rho^{3/2}\right)\, d\rho.
\end{aligned}$$

To solve $\sqrt{\rho} \leqslant 2-\rho^2$ for $\rho \geqslant 0$ notice that $\sqrt{\rho}$ is increasing with ρ while $2-\rho^2$ is decreasing. At $\rho = 1$ they concide, thus $\sqrt{\rho} \leqslant 2-\rho^2$ iff $0 \leqslant \rho \leqslant 1$. Therefore,

$$\lambda_3(D) = 2\pi \int_0^1 \left(2\rho - \rho^3 - \rho^{3/2}\right)\, d\rho = 2\pi \left[\rho^2 - \frac{\rho^4}{4} - \frac{\rho^{5/2}}{5/2}\right]_{\rho=0}^{\rho=1} = 2\pi\left(1 - \frac{1}{4} - \frac{2}{5}\right) = \frac{7}{10}\pi. \quad \square$$

iii) To compute the outward flux of $\vec{F} = (2x, 2y, 1)$ we apply the divergence theorem:

$$\int_{\partial D} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_D \operatorname{div} \vec{F}\, dxdydz = \int_D (2+2+0)\, dxdydz = 4\lambda_3(D) = \frac{14}{5}\pi.$$

To compute the component of this flux by the part of D on the surface $z = 2-(x^2+y^2)$, we need a specific parametrization. Here we can use the following standard parametrization:

$$\Phi(x,y) := (x, y, 2-x^2-y^2),\ \ (x,y) \in E = \{(x,y)\ :\ x^2+y^2 \leqslant 2\}.$$

Notice that

$$\vec{n}_\Phi = \frac{\partial_x\Phi \wedge \partial_y\Phi}{\|\partial_x\Phi \wedge \partial_y\Phi\|} = \frac{(1,0,-2x)\wedge(0,1,-2y)}{\|\cdots\|} = \frac{(2x,2y,1)}{\|(2x,2y,1)\|}.$$

In particular, from the third component of $\vec{n}_\Phi = 1$ we deduce that $\vec{n}_\Phi = \vec{n}_e$. Therefore, the flux of $\vec{F}$ through the surface $z = 2-x^2-y^2$ on D computed by using the parametrization Φ equals the component of the outward flux from D. This flux is

$$\begin{aligned}
\int_{x^2+y^2\leqslant 2} \det\begin{bmatrix} \vec{F} \\ \partial_x\Phi \\ \partial_y\Phi \end{bmatrix} dxdy &= \int_{x^2+y^2\leqslant 2} \det\begin{bmatrix} 2x & 2y & 1 \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} dxdy = \int_{x^2+y^2\leqslant 2} \left(4x^2 - \left(-4y^2-1\right)\right)\, dxdy \\
&= 4\int_{x^2+y^2\leqslant 2} (x^2+y^2)\, dxdy + 4\pi \\
&= 4\int_{0\leqslant\rho\leqslant\sqrt{2}} \rho^2\rho\, d\rho d\theta + 4\pi \overset{RF}{=} 4\cdot 2\pi \int_0^{\sqrt{2}} \rho^3\, d\rho + 4\pi \\
&= 4\pi\left(2\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=\sqrt{2}} + 1\right) = 12\pi.
\end{aligned}$$

Exercise 13. We need a parametrization of S. We could notice that

$$(x,y,z) \in S, \quad \Longleftrightarrow \quad z = (x^2+y^2)^{1/4},$$

hence look at S as the graph of a function $f(x,y) := (x^2+y^2)^{1/4}$. In alternative, we may notice that

$$(x,y,z) \in S, \quad \Longleftrightarrow \quad x^2+y^2 = z^4, \quad \Longleftrightarrow \quad x = z^2\cos\theta,\ y = z^2\sin\theta.$$

Thus

$$(x,y,z) = \Phi(z,\theta) = (z^2\cos\theta, z^2\sin\theta, z),\ (z,\theta) \in [0,1]\times[0,2\pi].$$

Therefore

$$\sigma_2(S) = \int_{[0,1]\times[0,2\pi]} \|\partial_z\Phi \wedge \partial_\theta\Phi\|\, dzd\theta.$$

Now,

$$\partial_z\Phi \wedge \partial_z\Phi = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2z\cos\theta & 2z\sin\theta & 1 \\ -z^2\sin\theta & z^2\cos\theta & 0 \end{bmatrix} = \left(-z^2\cos\theta, -z^2\sin\theta, 2z^3\right),$$

from which
$$\|\partial_z \Phi \wedge \partial_\theta \Phi\| = \sqrt{z^4 + 4z^6} = z^2\sqrt{1 + (2z)^2}.$$
Therefore
$$\sigma_2(S) = \int_{[0,1]\times[0,2\pi]} z^2\sqrt{1 + (2z)^2}\, dz \overset{RF}{=} 2\pi \int_0^1 z^2\sqrt{1 + (2z)^2}\, dz.$$
Now, since
$$\left[(1 + (2z)^2)^{3/2}\right]' = \frac{3}{2}(1 + (2z)^2)^{1/2} 2(2z) \cdot 2 = 6 \cdot 2z(1 + (2z)^2)^{1/2},$$
by parts,
$$\sigma_2(S) = \pi \left[\frac{1}{6}(1 + (2z)^2)^{3/2} z\Big|_{z=0}^{z=1} - \frac{1}{6}\int_0^1 (1 + (2z)^2)^{3/2}\, dz \right] = \pi \left[\frac{5^{3/2}}{6} - \frac{1}{6}\int_0^1 (1 + (2z)^2)^{3/2}\, dz \right]$$
Setting $2z = \sinh u$, $dz = \frac{1}{2}\cosh u\, du$, we have
$$\int (1 + (2z)^2)^{3/2}\, dz = \int (\cosh u)^3 \frac{1}{2}\cosh u\, du = \frac{1}{2}\int (\cosh u)^4\, du$$
A straightforward integration by parts leads to
$$\int (\cosh u)^4\, du = \frac{3}{8}u + \frac{1}{4}\sinh u \cosh u \left(\sinh^2 u + \frac{5}{2}\right) = \frac{3}{8}\sinh^{-1}(2z) + \frac{1}{2}z\sqrt{1 + 4z^2}\left(4z^2 + \frac{5}{2}\right)$$
Thus
$$\int_0^1 (1 + 4z^2)^{3/2}\, dz = \frac{3}{8}\sinh^{-1}(2) + \frac{1}{2}\sqrt{5}\left(4 + \frac{5}{2}\right).$$
From this we obtain the numerical value of $\sigma_2(S)$. □

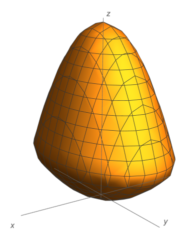
Exercise 14. i) We have a separable variable equation
$$y' = \frac{1}{t}(e^y - 1) = a(t) f(y).$$
Solutions are constant or non constant, in this case they are obtained by separation of variables. Constant solutions $y \equiv C$ are such that $0 = \frac{1}{t}(e^C - 1)$, iff $e^C - 1 = 0$, that is $C = 0$. Non constant solutions fulfils
$$\frac{y'}{e^y - 1} = \frac{1}{t}, \iff G(y)' = \frac{1}{t},$$
where
$$\begin{aligned} G(y) &= \int \tfrac{1}{e^y - 1}\, dy \overset{u=e^y,\ y=\log u,\ dy=du/u}{=} \int \tfrac{1}{u-1}\tfrac{du}{u} = \int \tfrac{1}{u(u-1)}\, du = \int \tfrac{1}{u-1} - \tfrac{1}{u}\, du = \log|u-1| - \log|u| \\ &= \log\left|1 - \tfrac{1}{u}\right| = \log|1 - e^{-y}|. \end{aligned}$$
Therefore, for a solution y,
$$(\log|1 - e^{-y}|)' = \frac{1}{t}, \iff \log|1 - e^{-y}| = \log|t| + c.$$
From this
$$|1 - e^{-y}| = |t|e^c = k|t|,\ \ k > 00, \iff 1 - e^{-y} = \pm k|t|, \iff e^{-y} = 1 \pm k|t|, \iff -y = \log(1 \pm k|t|),$$
and, since $k > 0$ means $\pm k \neq 0$, we have
$$y = -\log(1 + k|t|),\ k \neq 0.$$

Noticed that, for $k = 0$, we get back $y \equiv 0$, we can conclude that the general solution is

$$y(t) = -\log(1 + k|t|), \ k \in \mathbb{R}.$$

ii) Since $y(1) = -1$, imposing this to the general solution we obtain $-1 = -\log(1 + k)$, from which $1 + k = e$ and $k = e - 1$. □

Exercise 15. i) The statement is false. Indeed, by a rotation around the $z-$axis, the quantity $x^2 + y^2$ (which is the square of the distance to the $z-$axis) remains constant. Thus $x^2 + 2y^2 = k + y^2$. However, this last is not constant along the same rotation. Therefore, if a point $(x_0, y_0, z_0) \in D$, that is it fulfills the characteristic inequality that defines D, it is false that by rotating this point around the $z-$axis we will verify again the inequality.



ii) Volume:

$$\begin{aligned}
\lambda_3(D) &= \int_{x^2+2y^2 \leqslant z \leqslant 4-3(x^2+2y^2)} 1 \ dxdydz \overset{RF}{=} \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} \left(\int_{x^2+2y^2}^{4-3(x^2+2y^2)} dz \right) \ dxdy \\
&= \int_{x^2+2y^2 \leqslant 1} 4\left(1 - (x^2 + 2y^2)\right) \ dxdy \\
&\underset{\substack{x = \rho\cos\theta, \\ \sqrt{2}y = \rho\sin\theta,}}{=} \int_{\rho^2 \leqslant 1} 4(1 - \rho^2) \tfrac{1}{\sqrt{2}} \rho \ d\rho d\theta \overset{RF}{=} 2\pi \tfrac{4}{\sqrt{2}} \int_0^1 \rho - \rho^3 \ d\rho = \tfrac{8\pi}{\sqrt{2}} \left(\tfrac{1}{2} - \tfrac{1}{4} \right) = \tfrac{2\pi}{\sqrt{2}}.
\end{aligned}$$

iii) Let $\vec{F} = (4xz, -y^2, yz)$. The outward flux of $\vec{F}$ by D can be computed by the divergence thm: we have

$$\int_S \vec{F} \cdot \vec{n}_e \ d\sigma_2 = \int_D \operatorname{div} \vec{F} \ dxdydz = \int_D (4z - 2y + y) \ dxdydz = \int_D (4z - 3y) \ dxdydz.$$

To compute this integral we proceed as above for the volume. We could go straight to adapted cylindrical coordinates, obtaining

$$
\begin{aligned}
\int_D (4z-3y)\,dxdydz &= \frac{1}{\sqrt{2}}\int_{\rho^2\leqslant z\leqslant 4-3\rho^2}\left(4z-\frac{3}{\sqrt{2}}\rho\sin\theta\right)\rho\,d d\theta dz \\
&\overset{RF}{=} \frac{8\pi}{\sqrt{2}}\int_{\rho^2\leqslant z\leqslant 4-3\rho^2} z\rho\,dzd\rho - \frac{3}{2}\int_{\rho^2\leqslant z\leqslant 4-3\rho^2}\rho^2\underbrace{\left(\int_0^{2\pi}\sin\theta\,d\theta\right)}_{=0}\,d\rho dz \\
&= \frac{8\pi}{\sqrt{2}}\int_{\rho^2\leqslant z\leqslant 4-3\rho^2} z\rho\,dzd\rho \overset{RF}{=} \frac{8\pi}{\sqrt{2}}\int_{\rho^2\leqslant 4-3\rho^2}\rho\left(\int_{\rho^2}^{4-3\rho^2} z\,dz\right)d\rho \\
&= \frac{8\pi}{\sqrt{2}}\int_{\rho^2\leqslant 1}\rho\left[\frac{z^2}{2}\right]_{z=\rho^2}^{z=4-3\rho^2}d\rho = \frac{4\pi}{\sqrt{2}}\int_0^1\underbrace{\left((4-3\rho^2)^2-\rho^4\right)}_{=8\rho^4-24\rho^2+16}\,d\rho \\
&= \frac{4\pi}{\sqrt{2}}\left(8\left[\frac{\rho^5}{5}\right]_{\rho=0}^{\rho=1}-24\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1}+16\right) = \frac{4\pi}{\sqrt{2}}\frac{48}{5}.
\end{aligned}
$$

For the second part of point iii) we propose two alternative solutions.

First method. Let $S_1 = \partial D \cap \{z = x^2 + 2y^2$ and $S_2 = \partial D \cap \{z = 4 - 3(x^2 + 2y^2)\}$. The outward flux of $\vec{F}$ by ∂D is the sum of the fluxes through S_1 and S_2. We can campute one of these two deduce the other by difference. We compute the flux through S_1. To this aim we need a parametrization. This is easily offered by the analytical definition of S_1, $z = x^2 + 2y^2$, thus we may use the standard parametrization

$$\Phi(x, y) := (x, y, x^2 + 2y^2),\quad (x, y) \in E := \{(x, y)\ :\ x^2 + 2y^2 \leqslant 1\}.$$

Before computing the flux, we check if the normal $\vec{n}_\Phi$ is inward or outward. Recall that $\vec{n}_\Phi$ is the normalization of

$$\partial_x\Phi \wedge \partial_y\Phi = (1, 0, 2x) \wedge (0, 1, 4y) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 4y \end{bmatrix} = (-2x, -4y, 1).$$

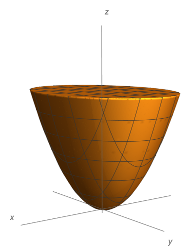
From this, and in particular from the third component, we see that $\vec{n}_\Phi$ is pointing inward. Thus, the component of outward flux from D on S_1 is

$$
\begin{aligned}
-\int_{S_1}\vec{F}\cdot\vec{n}_\Phi\,d\sigma_2 &= -\int_E \det\begin{bmatrix} 4x(x^2+2y^2) & -y^2 & y(x^2+2y^2) \\ 1 & 0 & 2x \\ 0 & 1 & 4y \end{bmatrix} dxdy \\
&= \int_{x^2+2y^2\leqslant 1} 8x^2(x^2+2y^2) + (-4y^2 - y(x^2+2y^2))\,dxdy
\end{aligned}
$$

The calculation is a bit long but elementary. We leave to the reader to complete it.

Second Method. Instead of completing the calculation in this way, we show an alternative way that sometimes (and in certain specific situations) might simplify the task. We may look at S_1 also as part of boundary of another domain $\widetilde{D}$:

$$\widetilde{D} := \{(x, y, z)\ :\ x^2 + 2y^2 \leqslant z \leqslant 1\}.$$



In this case $\partial\widetilde{D} = S_1 \cup S_3$ where S_3 is the ellypsis $S_3 := \partial\widetilde{D} \cap \{z = 1\} = \{(x, y, 1) \ : \ x^2 + 2y^2 \leqslant 1\} =: E_3$. The outward flux of $\vec{F}$ from $\widetilde{D}$ is

$$\int_{\partial\widetilde{D}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{S_1} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{S_3} \vec{F} \cdot \vec{n}_e \, d\sigma_2$$

Then

$$\int_{S_1} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{S_3} \vec{F} \cdot \vec{n}_e \, d\sigma_2 - \int_{\partial\widetilde{D}} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

Now, on S_3 clearly $\vec{n}_e = (0, 0, 1)$, thus

$$\int_{S_3} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{S_3} (4xz, -y^2, yz) \cdot (0, 0, 1) \, d\sigma_2 = \int_{S_3} yz \, d\sigma_2 \overset{z=1 \ on \ S_3}{=} \int_{S_3} y \, d\sigma_3.$$

By the above description, $S_3 = \Phi(x, y) = (x, y, 1)$, $(x, y) \in E_3$, thus

$$\int_{S_3} y \, d\sigma_3 = \int_{E_3} y \|\partial_x\Phi \wedge \partial_y\Phi\| \, dxdy.$$

Now,

$$\partial_x\Phi \wedge \partial_y\Phi = (1, 0, 0) \wedge (0, 1, 0) = (0, 0, 1),$$

so

$$\int_{S_3} y \, d\sigma_3 = \int_{E_3} y \, dxdy = \int_{x^2+2y^2\leqslant 1} y \, dxdy \overset{\substack{x = \rho\cos\theta, \\ \sqrt{2}y = \rho\sin\theta,}}{=} \frac{1}{\sqrt{2}} \int_{\rho^2 \leqslant 1} \frac{1}{\sqrt{2}} \rho\sin\theta \, \rho \, d\rho d\theta \overset{RF}{=} \frac{1}{2} \int_0^1 \rho^2 \underbrace{\left(\int_0^{2\pi} \sin\theta \, d\theta \right)}_{=0} d\rho = 0.$$

Finally, by the divergence thm (applied here on domain $\widetilde{D}$), we have

$$\int_{\partial\widetilde{D}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\widetilde{D}} \operatorname{div} \vec{F} \, dxdydz = \int_{\widetilde{D}} (4z - 3y) \, dxdydz \overset{adapt.\ cyl.\ coords}{=} \frac{1}{\sqrt{2}} \int_{\rho^2 \leqslant z \leqslant 1} \left(4z - \frac{3}{\sqrt{2}} \rho\sin\theta \right) \rho \, d\rho dz d\theta.$$

As above, the second term vanishes, thus we need just to compute

$$\int_{\rho^2 \leqslant z \leqslant 1} z\rho \, d\rho d\theta dz \overset{RF}{=} 2\pi \int_0^1 \left(\int_{\rho^2}^1 \rho z \, dz \right) d\rho = 2\pi \int_0^1 \rho \left[\frac{z^2}{2} \right]_{z=\rho^2}^{z=1} d\rho = \pi \int_0^1 \rho - \rho^5 \, d\rho = \frac{\pi}{3}.$$

Conclusion:

$$\int_{S_1} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = -\frac{4\pi}{3\sqrt{2}}.$$

The other component can be computed by difference. □

Exercise 16. To compute the area of the surface, we first need a parametrization of S. Since

$$S := \{(x, y, z) \in \mathbb{R}^3 \ : \ z = x^2 + 2y^2, 0 \leqslant z \leqslant 1\}$$

we can use the standard "graph parametrization", $z = f(x, y) := x^2 + 2y^2$, defined on $E := \{(x, y) \ : \ 0 \leqslant x^2 + 2y^2 \leqslant 1\}$. Then

$$\begin{aligned}
\sigma_2(S) &= \int_E \sqrt{1 + \|\nabla f\|^2}\, dxdy = \int_{x^2+2y^2\leqslant 1} \sqrt{1 + \|(2x, 4y)\|^2}\, dxdy = \int_{x^2+2y^2\leqslant 1} \sqrt{1 + 4(x^2 + 4y^2)}\, dxdy \\
&\overset{\substack{x = \rho\cos\theta, \\ \sqrt{2}y = \rho\sin\theta,}}{=} \frac{1}{\sqrt{2}} \int_{\rho^2\leqslant 1} \sqrt{1 + 4\rho^2}\rho\, d\rho d\theta \\
&\overset{RF}{=} \frac{2\pi}{\sqrt{2}} \int_0^1 (1 + 4\rho^2)^{1/2}\rho\, d\rho = \frac{\pi}{6\sqrt{2}} \left[(1 + 4\rho^2)^{3/2}\right]_{\rho=0}^{\rho=1} = \frac{\pi}{6\sqrt{2}} \left(5^{3/2} - 1\right). \qquad \square
\end{aligned}$$

Exercise 17. i) The general integral is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t),$$

where (w_1, w_2) is a fundamental system of solutions for the homogeneous equation $y'' - 2y' + y = 0$ and u is a particular solution of the equation. The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0, \quad \Longleftrightarrow \quad (\lambda - 1)^2 = 0, \quad \Longleftrightarrow \quad \lambda_{1,2} = 1.$$

Therefore, the fundamental system of solutions is $w_1 = e^t$, $w_2 = te^t$. To compute the particular solution u we apply the Lagrange formula

$$u(t) = \left(-\int \frac{w_2}{W} f\, dt\right) w_1 + \left(\int \frac{w_1}{W} f\, dt\right) w_2,$$

where W is the wronskian

$$W = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} = (t + 1)e^{2t} - te^{2t} = e^{2t},$$

and $f = f(t) = e^{2t}$. Thus

$$u(t) = \left(-\int \frac{te^t}{e^{2t}} e^{2t}\, dt\right) e^t + \left(\int \frac{e^t}{e^{2t}} e^{2t}\, dt\right) (te^t) = -\left(te^t - \int e^t\, dt\right) e^t + e^t te^t = e^{2t}.$$

Conclusion: the general integral is

$$y(t) = c_1 e^t + c_2 te^t + e^{2t}, \ c_1, c_2 \in \mathbb{R}.$$

ii) To solve the Cauchy problem we impose the initial conditions $y(0) = 1$ and $y'(0) = 0$ to the general integral. First notice that

$$y' = c_1 e^t + c_2(t + 1)e^t + 2e^{2t},$$

thus

$$\begin{cases} y(0) = 1, \\ y'(0) = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} c_1 + 1 = 1, \\ c_1 + c_2 + 2 = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} c_1 = 0, \\ c_2 = -2, \end{cases}$$

and the solution is $y(t) = -2te^t + e^{2t}$.

iii) Again, we impose the passage conditions

$$\begin{cases} c_1 + 1 = 0, \\ c_1 e + c_2 e + e^2 = a, \end{cases} \iff \begin{cases} c_1 = -1, \\ c_2 = \frac{a-e^2+e}{e}. \end{cases}$$

We conclude that: for every $a \in \mathbb{R}$ there exists a unique solution to the proposed problem. □

Exercise 18. i) D is the intersection of the sphere $\{x^2 + y^2 + z^2 \leqslant 4\}$ centred at $(0, 0, 0)$ with radius 2, and the region above the paraboloid $z = 1 - (x^2 + y^2)$. Since both sets are described through constraint that depend on (x, y) through $x^2 + y^2$, D is ivariant by rotations around the $z-$axis. We can start drawing the section of D on the yz plane, that is

$$D \cap \{x = 0\} = \{(0, y, z) \ : \ y^2 + z^2 \leqslant 4, \ z \geqslant 1 - y^2\}$$

hence we can rotate this section around the $z-$ axis to get D. The result is shown by the following figure:

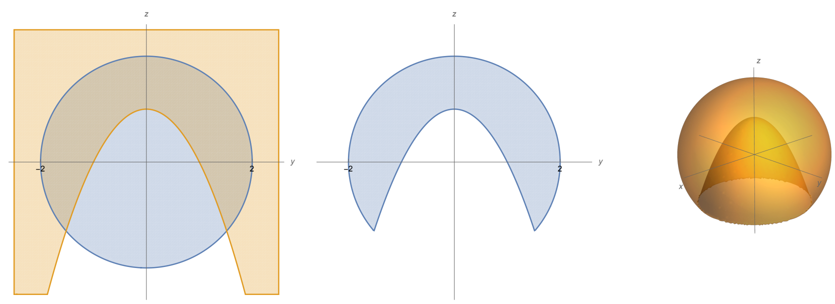


FIGURE 2. From left to right: plots of $y^2 + z^2 \leqslant 4$ and $z \geqslant 1 - y^2$, plot of $D \cap \{x = 0\}$, plot of D.

ii) We have

$$\lambda_3(D) = \int_D 1 \, dxdydz \overset{cyl.\ coords.}{=} \int_{\rho^2+z^2\leqslant 4, \ z\geqslant 1-\rho^2} \rho \, d\rho d\theta dz \overset{RF}{=} 2\pi \int_{\rho^2+z^2\leqslant 4, \ z\geqslant 1-\rho^2} \rho \, d\rho dz$$

To apply RF notice that

$$\rho^2 + z^2 \leqslant 4, \ z \geqslant 1 - \rho^2, \quad \iff \quad 1 - z \leqslant \rho^2 \leqslant 4 - z^2.$$

When $1 - z < 0$, that is $z > 1$, the inequality writes

$$\rho^2 \leqslant 4 - z^2, \iff \begin{cases} z^2 \leqslant 4, \\ 0 \leqslant \rho \leqslant \sqrt{4 - z^2}, \end{cases} \iff \begin{cases} 1 < z < 2, \\ 0 \leqslant \rho \leqslant \sqrt{4 - z^2}, \end{cases}$$

while, when $z \leqslant 1$, the inequality writes,

$$\begin{cases} 1 - z \leqslant 4 - z^2, \\ \sqrt{1 - z} \leqslant \rho \leqslant \sqrt{4 - z^2} \end{cases} \iff \begin{cases} \frac{1-\sqrt{13}}{2} \leqslant z \leqslant 1, \\ \sqrt{1 - z} \leqslant \rho \leqslant \sqrt{4 - z^2}. \end{cases}$$

Therefore

$$\lambda_3(D) \overset{RF}{=} 2\pi\left(\int_{\frac{1-\sqrt{13}}{2}}^{1}\int_{\sqrt{1-z}}^{\sqrt{4-z^2}} \rho\, d\rho\, dz + \int_1^2\int_0^{\sqrt{4-z^2}} \rho\, d\rho\, dz\right) = 2\pi\left(\int_{\frac{1-\sqrt{13}}{2}}^{1}\left[\frac{\rho^2}{2}\right]_{\rho=\sqrt{1-z}}^{\rho=\sqrt{4-z^2}} dz + \int_1^2\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{4-z^2}} dz\right)$$

$$= \pi\left(\int_{\frac{1-\sqrt{13}}{2}}^{1}\left(4-z^2-(1-z)\right) dz + \int_1^2 4-z^2\, dz\right) = \pi\left(\int_{\frac{1-\sqrt{13}}{2}}^{2} 4-z^2\, dz + \int_{\frac{1-\sqrt{13}}{2}}^{1} z-1\, dz\right)$$

$$= \pi\left(6+2\sqrt{13} - \left[\frac{z^3}{3}\right]_{z=\frac{1-\sqrt{13}}{2}}^{z=2} + \left[\frac{z^2}{2}\right]_{z=\frac{1-\sqrt{13}}{2}}^{z=1} - \frac{1+\sqrt{13}}{2}\right)$$

$$= \pi\left(6+2\sqrt{13} - \tfrac{1}{3}\left(1-\left(\tfrac{1-\sqrt{13}}{2}\right)^3\right) + \tfrac{1}{2}\left(1-\left(\tfrac{1-\sqrt{13}}{2}\right)^2\right) - \tfrac{1+\sqrt{13}}{2}\right).$$

iii) To compute the outward flux we apply the divergence theorem. We have

$$\int_{\partial D} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_D \operatorname{div}\vec{F}\, dxdydz = \int_D 3\, dxdydz = 3\lambda_3(D).$$

To compute the component of this flux on $\partial D \cap \{x^2+y^2+z^2=4\}$ we compute the component along $\partial D \cap \{z = 1-(x^2+y^2)\}$. The standard parametrization is

$$\Phi(x,y) := (x, y, 1-(x^2+y^2)),\ (x,y)\in E = \left\{(x,y)\ :\ x^2+y^2 \leqslant \frac{1+\sqrt{13}}{2}\right\}.$$

Let's compute the normal $\vec{n}_\Phi$. We have

$$\partial_x\Phi \wedge \partial_y\Phi = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} = (2x, 2y, 1),$$

thus

$$\vec{n}_\Phi \parallel (2x, 2y, 1)$$

from which we see that $\vec{n}_\Phi$ is pointing inward for D, that is $\vec{n}_\Phi = -\vec{n}_e$. Therefore

$$\int_{\partial D\cap\{z=1-(x^2+y^2)\}} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_E \det\begin{bmatrix} \nabla\vec{F} \\ \partial_x\Phi \\ \partial_y\Phi \end{bmatrix} dxdy = \int_E \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{bmatrix} dxdy = \int_E 2x+2y\, dxdy$$

$$= 2\int_{x^2+y^2\leqslant\frac{1+\sqrt{13}}{2}} (x+y)\, dxdy \overset{pol.\ coords}{=} 2\int_{0\leqslant\rho\leqslant\sqrt{\frac{1+\sqrt{13}}{2}}} \rho(\cos\theta+\sin\theta)\, d\rho d\theta$$

$$\overset{RF}{=} 2\int_0^{\sqrt{\frac{1+\sqrt{13}}{2}}} \rho \underbrace{\int_0^{2\pi} (\cos\theta+\sin\theta)\, d\theta}_{=0} d\rho = 0. \quad \square$$

Exercise 19. An immediate parametrization of S is of course

$$S = \Phi(E),\ \ \Phi(x,y) := (4-(y^2+z^2), y, z),\ (y,z)\in E := \left\{(y,z)\ :\ 2\leqslant y^2+z^2\leqslant 3\right\}.$$

Then

$$\sigma_2(S) = \int_E \|\partial_y \Phi \wedge \partial_z \Phi\| \, dydz.$$

Notice that

$$\partial_y \Phi \wedge \partial_z \Phi = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2y & 1 & 0 \\ -2z & 0 & 1 \end{bmatrix} = (1, 2y, 2z), \implies \|\partial_y \Phi \wedge \partial_z \Phi\| = \sqrt{1 + 4(y^2 + z^2)}.$$

Therefore

$$\begin{aligned} \sigma_2(S) &= \int_{2 \leqslant y^2+z^2 \leqslant 3} \sqrt{1 + 4(y^2 + z^2)} \, dydz \stackrel{pol.\ coords}{=} \int_{2 \leqslant \rho^2 \leqslant 3} \sqrt{1 + 4\rho^2}\rho \, d\rho d\theta \stackrel{RF}{=} 2\pi \int_{\sqrt{2}}^{\sqrt{3}} (1 + 4\rho^2)^{1/2} \rho \, d\rho \\ &= \frac{\pi}{6} \left[\left(1 + 4\rho^2\right)^{3/2} \right]_{\rho=\sqrt{2}}^{\rho=\sqrt{3}} = \frac{\pi}{6} \left(\sqrt{13} - 3 \right). \quad \square \end{aligned}$$