Calculus II

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Contents

CHAPTER 1

Euclidean Space $\mathbb{R}^d$

In a large part of this course, we will do Analysis in a multidimensional context,

$$\mathbb{R}^d := \{(x_1, \dots, x_d) \; : \; x_i \in \mathbb{R}, \; i = 1, \dots, d\}.$$

The basic tool of Analysis, the concept of *limit* with its applications to continuity, differentiability, integrability, is the protagonist of this Chapter. To catch the idea, let us recall we say that a sequence $(x_n) \subset \mathbb{R}$ has $\lim_{n \to +\infty} x_n = \ell \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \; \exists N \in \mathbb{N} \; : \; |x_n - \ell| \leqslant \varepsilon, \; \forall n \geqslant N.$$

The number $|x_n - \ell|$ represents the distance between x_n and ℓ and the previous property can be interpreted as follows: *the distance between x_n and the limit ℓ becomes arbitrarily small provided n is large enough*. If now $(\vec{x}_n) \subset \mathbb{R}^d$ and we want to say $\lim_n \vec{x}_n = \vec{\ell} \in \mathbb{R}^d$ we need something similar to the modulus to measure the distance between $\vec{x}_n$ and $\vec{\ell}$. This leads to the concept of **norm**, which is the starting point of our story.

Chapter requirements: concept of vector space, limit for a function of one real variable, basic facts on continuous functions of one real variable.

1.1. Euclidean norm

We start recalling that $\mathbb{R}^d$ is a *vector space on* $\mathbb{R}$ with operations of *sum* and *product* defined as

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d), \quad \lambda(x_1, \dots, x_d) := (\lambda x_1, \dots, \lambda x_d).$$

Given any two vectors $\vec{x} = (x_1, \dots, x_d)$, $\vec{y} = (y_1 \dots, y_d)$, there is a natural way to measure the distance between $\vec{x}$ and $\vec{y}$. For ease of simplicity, consider the case of dimension $d = 2$ (see the figure). According to Pythagorean Theorem, the length of the segment joining $\vec{x}$ to $\vec{y}$ is

$$\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This idea can be introduced in the general $\mathbb{R}^d$: given $\vec{x} = (x_1, \dots, x_d)$ and $\vec{y} = (y_1, \dots, y_d)$, then

$$\mathrm{dist}(\vec{x}, \vec{y}) = \sqrt{\sum_{j=1}^{d} (x_j - y_j)^2}.$$

Since $\vec{x} - \vec{y} = (x_1 - y_1, \dots, x_d, -y_d)$, previous formula suggests the following

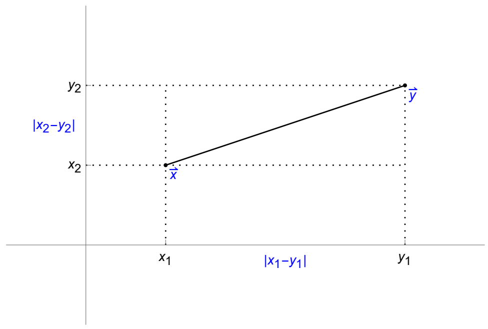


FIGURE 1. Euclidean distance

Definition 1.1.1

Given $\vec{x} = (x_1, \dots, x_d) \in \mathbb{R}$, we call **euclidean norm of** x the quantity

$$\|\vec{x}\| := \sqrt{\sum_{j=1}^{d} x_j^2}.$$

The norm plays the same role of the modulus in $\mathbb{R}$. Precisely

Proposition 1.1.2

Euclidean norm fulfils the following properties:

i) **positivity**: $\|\vec{x}\| \geqslant 0$, $\forall \vec{x} \in \mathbb{R}^d$;
ii) **vanishing**: $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0} := (0, \dots, 0)$.
iii) **homogeneity**: $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$, $\forall \lambda \in \mathbb{R}$, $\forall \vec{x} \in \mathbb{R}^d$.
iv) **triangular inequality**: $\|\vec{x}+\vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|$, $\forall\, \vec{x}, \vec{y} \in \mathbb{R}^d$.

PROOF. Positivity is evident. Let us check the vanishing:

$$\|\vec{x}\| = 0, \iff \sum_{j=1}^{d} x_j^2 = 0, \iff x_j^2 = 0,\ \forall i, \iff x_j = 0,\ \forall j = 1, \dots, d.$$

Homogeneity is straightforward:

$$\|\lambda\vec{x}\| = \sqrt{\sum_j (\lambda x_j)^2} = \sqrt{\lambda^2 \sum_j x_j^2} = |\lambda| \sqrt{\sum_j x_j^2} = |\lambda|\,\|\vec{x}\|.$$

Finally, the triangular inequality: for convenience let us square everything and notice that

$$\|\vec{x}+\vec{y}\|^2 = \sum_j (x_j + y_j)^2 = \sum_j (x_j^2 + y_j^2 + 2x_j y_j) = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\sum_j x_j y_j = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \vec{x}\cdot\vec{y},$$

where $\vec{x}\cdot\vec{y}$ is the scalar product of $\mathbb{R}^d$. At this point we need the

Lemma 1.1.3: Cauchy–Schwarz inequality

$$\vec{x} \cdot \vec{y} \leqslant \|\vec{x}\|\|\vec{y}\|, \ \forall \vec{x}, \vec{y} \in \mathbb{R}^d. \tag{1.1.1}$$

Proof. (Lemma) Excluding the trivial cases where $\|\vec{x}\| = 0$ or $\|\vec{y}\| = 0$ we may assume $\|\vec{x}\|, \|\vec{y}\| \neq 0$. The formula (1.1.1) is equivalent to

$$\sum_j \frac{x_j}{\|\vec{x}\|} \frac{y_j}{\|\vec{y}\|} \leqslant 1.$$

Notice now that

$$ab \leqslant \frac{1}{2}(a^2 + b^2), \ \left(\iff 2ab \leqslant a^2 + b^2, \ \iff \ (a-b)^2 \geqslant 0 \right).$$

Applying this inequality to $a = \frac{x_j}{\|\vec{x}\|}$ and $b = \frac{y_j}{\|\vec{y}\|}$, and summing on j, we have

$$\sum_j \frac{x_j}{\|\vec{x}\|} \frac{y_j}{\|\vec{y}\|} \leqslant \frac{1}{2} \sum_j \left(\frac{x_j^2}{\|\vec{x}\|^2} + \frac{y_j^2}{\|\vec{y}\|^2} \right) = \frac{1}{2} \left(\frac{\|\vec{x}\|^2}{\|\vec{x}\|^2} + \frac{\|\vec{y}\|^2}{\|\vec{y}\|^2} \right) = 1. \quad \square$$

By Cauchy–Schwarz,

$$\|\vec{x} + \vec{y}\|^2 \leqslant \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\|\vec{y}\| = \left(\|\vec{x}\| + \|\vec{y}\|\right)^2, \ \iff \ \|\vec{x} + \vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|. \quad \square$$

From the norm, we define the concept of *limit for a sequence* and, later, the limit of a function:

Definition 1.1.4

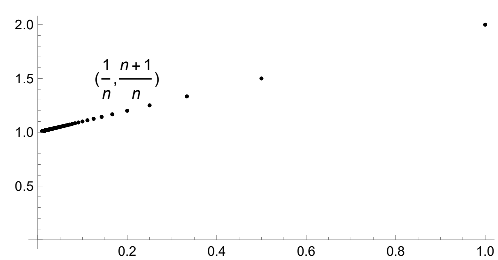
Let $(\vec{x}_n) \subset \mathbb{R}^d$. We say that

$$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \ \iff \ \|\vec{x}_n - \vec{\ell}\| \longrightarrow 0.$$

Example 1.1.5. *Show that* $\left(\frac{1}{n}, \frac{n+1}{n}\right) \longrightarrow (0, 1)$.

Sol. — According to the definition

$$\left\| \left(\frac{1}{n}, \frac{n+1}{n} \right) - (0,1) \right\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{n+1}{n} - 1\right)^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} \longrightarrow 0. \quad \square$$



It is not difficult to prove that

Proposition 1.1.6: component-wise convergence

Let $\vec{x}_n = (x_{n,1}, \dots, x_{n,d})$. Then

$$\vec{x}_n \longrightarrow \vec{\ell} = (\ell_1, \dots, \ell_d), \iff x_{n,j} \longrightarrow \ell_j, \ \forall j = 1, \dots, d.$$

Example 1.1.7. *Is* $\vec{x}_n := \left(\frac{1}{n}, \frac{1-n}{n}, \sin n\right)$ *convergent in* $\mathbb{R}^3$*?*

Sol. — The components of $\vec{x}_n$ are $\frac{1}{n} \longrightarrow 0$, $\frac{1-n}{n} = \frac{1}{n} - 1 \longrightarrow -1$ and $\sin n$, which has no limit. We conclude that $\vec{x}_n$ is not convergent in $\mathbb{R}^3$. □

A little bit of care is needed for $\vec{x}_n \longrightarrow \infty$ because, differently by $\mathbb{R}$, there are not $+\infty$ and $-\infty$:

Definition 1.1.8

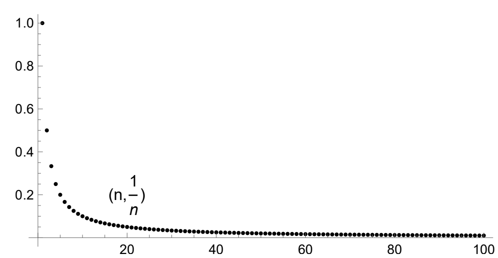
Let $(\vec{x}_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that

$$\vec{x}_n \longrightarrow \infty_d, \iff \|\vec{x}_n\| \longrightarrow +\infty.$$

Example 1.1.9. *Show that* $(n, \frac{1}{n}) \longrightarrow \infty_2$.

Sol. — We have

$$\left\|\left(n, \frac{1}{n}\right)\right\| = \sqrt{n^2 + \frac{1}{n^2}} \longrightarrow +\infty. \quad \square$$



Remark 1.1.10. *In particular, say that* $\vec{x}_n \longrightarrow \infty_d$ *iff* $x_{n,j} \longrightarrow \infty$ *is false.* □

1.2. Limit of a function

In this section, we want to define the notion of limit,

$$\lim_{x \to x_0} \vec{F}(\vec{x}) = \vec{\ell}.$$

Here

$$\boxed{\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m.}$$

As for one variable limit, limit makes sense only when $\vec{x}$ approaches an *accumulation point* of the domain D. The definition of accumulation point is similar to the case of $\mathbb{R}$:

Definition 1.2.1

Let $D \subset \mathbb{R}^d$. We say that

- $\vec{x}_0 \in \mathbb{R}^d$ is **accumulation point for** D if $\exists(\vec{x}_n) \subset D \backslash \{\vec{x}_0\}$ such that $\vec{x}_n \longrightarrow \vec{x}_0$;
- ∞_d is **accumulation point for** D if $\exists(\vec{x}_n) \subset D$ such that $\vec{x}_n \longrightarrow \infty_d$.

The set of all accumulation points of D is denoted by $\mathrm{Acc}(D)$.

We are now ready for the

Definition 1.2.2

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ and $\vec{x}_0 \in \mathrm{Acc}(D)$. We say that

$$\lim_{\vec{x} \to \vec{x}_0} \vec{F}(\vec{x}) = \vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}, \iff \vec{F}(\vec{x}_n) \longrightarrow \vec{\ell},\ \forall (\vec{x}_n) \subset D \backslash \{\vec{x}_0\},\ \vec{x}_n \longrightarrow \vec{x}_0. \tag{1.2.1}$$

This Definition has the advantage of covering all the possibilities: limits at a finite point (when $\vec{x}_0 \in \mathbb{R}^d$), at infinite (when $\vec{x}_0 = \infty_d$) and finite limits (when $\vec{\ell} \in \mathbb{R}^m$) or infinite limits ($\vec{\ell} = \infty_m$).

Compared to the limits of a single variable, the calculation of limits for vector-valued functions of a vector variable can be extremely complicated. For simplicity, here we will limit ourselves to the case of *numerical functions of the vector variable*, that is,

$$\boxed{f = f(\vec{x})\ :\ D \subset \mathbb{R}^d \longrightarrow \mathbb{R}.}$$

1.2.1. Sections. A natural idea is to reduce the problem of computing a limit in several variables into a limit in a unique variable. Is it possible? And how? A way to do this is *restricting* f to a *curve* that approaches point $\vec{x}_0$.

Definition 1.2.3: Curve

A **curve** is a continuous function

$$\boxed{\vec{\gamma} = \vec{\gamma}(t)\ :\ I \subset \mathbb{R} \longrightarrow \mathbb{R}^d,\ \ \gamma \in \mathscr{C}(I).}$$

We say that γ is in D (notation $\gamma \subset D$) if $\gamma(t) \in D$ for every $t \in I$.

Evaluating f on γ, that is computing $f(\gamma(t))$, means "sectioning" f along γ. Next result provides a relation between the limit of f and limit of its sections.

Proposition 1.2.4

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be such that

$$\exists \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = \ell \in \mathbb{R} \cup \{\pm\infty\}$$

Then

$$\lim_{t\to t_0} f(\vec{\gamma}(t)) = \ell,\ \forall \gamma \text{ curve},\ \gamma \subset D\backslash\{\vec{x}_0\},\ :\ \gamma(t) \overset{t\longrightarrow t_0}{\longrightarrow} x_0.$$

Proof. It is just an application of the definition. Take $t_n \longrightarrow t_0$, $t_n \neq t_0$. Then

$$\vec{\gamma}(t_n) \longrightarrow \vec{x}_0,$$

and since $\vec{\gamma}(t_n) \neq \vec{x}_0$, we have

$$f(\vec{\gamma}(t_n)) \longrightarrow \ell. \quad \square$$

In practice, **if** *f has limit ℓ for $\vec{x} \longrightarrow \vec{x}_0$*, **then** *every section of f going to point $\vec{x}_0$ has the same limit*. Actually, this fact is useful to **disprove** that f has a limit. Indeed, if one proves that there are two sections along which

$$\lim_{t\to t_0} f(\vec{\gamma}_1(t)) \neq \lim_{t\to t_0} f(\vec{\gamma}_2(t))$$

then $\lim_{x\to x_0} f(x)$ cannot exists.

Example 1.2.5. *Show that*

$$\lim_{(x,y)\to 0_2} \frac{xy}{x^2+y^2}$$

does not exist.

Sol. — Let

$$f(x,y) = \frac{xy}{x^2+y^2},\ (x,y) \in D = \mathbb{R}^2\backslash\{(0,0)\}.$$

Let us check what happens along the two sections of the two axes. These are given by

$$\gamma_1(t) = (t,0), \quad \gamma_2(t) = (0,t).$$

Clearly

$$\vec{\gamma}_1(t) \longrightarrow (0,0),\ \ \vec{\gamma}_2(t) \longrightarrow (0,0),\ \ \text{when } t \longrightarrow 0.$$

We have

$$f(\vec{\gamma}_1(t)) = f(t,0) \equiv 0 \longrightarrow 0,\ \ f(\vec{\gamma}_2(t)) = f(0,t) \equiv 0 \longrightarrow 0,\ \ \text{when } t \longrightarrow 0.$$

Is this enough to conclude that the limit exists (and, in the case, it equals 0)? A big NO! This because we checked only two of the infinitely many sections. Let consider a new section, that is a point moving along a straight line $y = mx$. The curve describing this is simply

$$\vec{\gamma}_3(t) := (t, mt),\ m \in \mathbb{R}.$$

Notice that the corresponding section of f is

$$f(\vec{\gamma}_3(t)) = f(t,mt) = \frac{mt^2}{t^2+m^2t^2} = \frac{m}{1+m^2} \longrightarrow \frac{m}{1+m^2},\ \text{as } t \longrightarrow 0.$$

We conclude that the limit does not exist. $\square$

Next example shows that one might have limits along all straight sections, yet this is not sufficient to ensure existence of the limit.

EXAMPLE 1.2.6. *Show that*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

does not exist.

SOL. — Let

$$f(x,y) = \frac{xy^2}{x^2+y^4},\ (x,y)\in D = \mathbb{R}^2\backslash\{(0,0)\}.$$

The sections along the axes are $f(t,0)\equiv 0 \longrightarrow 0$ and $f(0,t)\equiv 0\longrightarrow 0$ both when $t\longrightarrow 0$. This says: **if the limit exists, it must be equal to** 0. Now if we take a section along the line $y = mx$,

$$f(t,mt) = \frac{m^2t^3}{t^2+m^2t^4} = \frac{m^2t}{1+m^2t^2} \longrightarrow 0,\ \text{as } t\longrightarrow 0.$$

So apparently again no contradictions! But when we consider the line $x = ay^2$ we have

$$f(at^2,t) = \frac{at^2t^2}{a^2t^4+t^4} = \frac{a}{a^2+1} \longrightarrow \frac{a}{a^2+1},\ \text{as } t\longrightarrow 0.$$

This is different from 0 if $a\neq 0$: so we have found a family of curves on which the limit of f exists but is different on any family: we deduce that the limit doesn't exists. □

EXAMPLE 1.2.7. *Show that*

$$\lim_{(x,y)\to\infty_2}\left(x^2+y^2-4xy\right)$$

does not exist.

SOL. — Let $f(x,y) := x^2+y^2-4xy$. Sections along the axes are $f(t,0) = t^2$, $f(0,t) = t^2$. Clearly the points $(t,0), (0,t)$ go to ∞_2 if $t\longrightarrow\pm\infty$. In any case $f(t,0), f(0,t)\longrightarrow +\infty$. So the candidate to be the eventual limit is $+\infty$. However, along the line $y = ax$,

$$f(t,at) = t^2+a^2t^2-4at^2 = (1+a^2-4a)t^2.$$

If we chose a in such a way that $1+a^2-4a = 0$, that is $a = \frac{4\pm\sqrt{12}}{2} = 2\pm\sqrt{3}$ we see that $f(t,at)\equiv 0\longrightarrow 0$ for $t\longrightarrow\pm\infty$. We conclude the limit does not exist. □

In conclusion, sections may be used

- *to guess the possible candidate limit*;
- *to exclude existence of the limit*.

1.2.2. Methods of calculus. We now introduce a technique which is sometimes useful to prove that limit exists. We start with an example, hence we draw the general rule:

EXAMPLE 1.2.8. *Compute*

$$\lim_{(x,y)\to 0_2}\frac{xy^2}{x^2+y^2}.$$

Sol. — We start looking at f along standard sections. We have, $f(x,0)\equiv 0 \longrightarrow 0$ for $x\longrightarrow 0$, so if the limit exists it must be 0. This is confirmed by the $y-$axis section since $f(0,y)\equiv 0\longrightarrow 0$ for $y\longrightarrow 0$. More in general, along $y=mx$ we have

$$f(x,mx)=\frac{xm^2x^2}{x^2+m^2x^2}=x\frac{m^2}{1+m^2}\longrightarrow 0,\ \ x\longrightarrow 0.$$

This says that *if* limit exists, it must be equal to 0. However, as we know, this does not prove yet existence. Let us give a look to f by using different coordinates respect to Cartesian ones: in polar coordinates

$$\begin{cases} x=\rho\cos\theta, \\ \\ y=\rho\sin\theta, \end{cases}$$

we have

$$f(\rho\cos\theta,\rho\sin\theta)=\frac{\rho^3(\cos\theta)(\sin\theta)^2}{\rho^2}=\rho(\cos\theta)(\sin\theta)^2,$$

thus

$$|f(\rho\cos\theta,\rho\sin\theta)|\leqslant|\rho(\cos\theta)(\sin\theta)^2|\leqslant\rho.$$

Returning to euclidean coordinates this last says that

$$|f(x,y)|\leqslant\|(x,y)\|.$$

Hence, if $(x,y)\longrightarrow 0_2$, that is $\|(x,y)\|\longrightarrow 0$ then $|f(x,y)|\longrightarrow 0$, that is $f(x,y)\longrightarrow 0$. □

What is the general argument used here? Basically, we obtained a bound

$$|f(x,y)|\leqslant\|(x,y)\|.$$

Since $(x,y)\longrightarrow\vec{0}$ is equivalent to say $\|(x,y)\|\longrightarrow 0$, by Police theorem we conclude that $f(x,y)\longrightarrow 0$. Notice that the important fact is that $|f(x,y)|$ is controlled by a quantity that goes to 0 when $\|(x,y)\|\longrightarrow 0$. In other words, if

$$|f(x,y)|\leqslant\phi(\|(x,y)\|),$$

where $\phi=\phi(\rho)\longrightarrow 0$ when $\rho\longrightarrow 0$, the conclusion would be the same. In general, if $f=f(\vec{x})$, is such that

$$\boxed{|f(\vec{x})|\leqslant\phi(\|\vec{x}\|),\ \ \text{with}\ \phi=\phi(\rho)\longrightarrow 0,\ \rho\longrightarrow 0}$$

the conclusion is

$$\lim_{\vec{x}\longrightarrow\vec{0}} f(\vec{x})=0.$$

Here is another example:

Example 1.2.9. *Compute*

$$\lim_{(x,y,z)\to 0_3}\frac{\sin(xyz)}{x^2+y^2+z^2}.$$

Sol. — Let $f(x,y,z):=\frac{\sin(xyz)}{x^2+y^2+z^2}$ defined on its natural domain $D=\mathbb{R}^3\backslash\{0_3\}$. The sections on the axes $f(x,0,0)=f(0,y,0)=f(0,0,z)\equiv 0$, so the eventual candidate to be the limit is 0. Using spherical coordinates

$$\begin{cases} x=\rho\cos\theta\sin\varphi, \\ y=\rho\sin\theta\sin\varphi, \\ z=\rho\cos\varphi, \end{cases}$$

we have

$$f(\rho\cos\theta\sin\varphi,\rho\sin\theta\sin\varphi,\rho\cos\varphi)=\frac{\sin\left(\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)\right)}{\rho^2}.$$

Recalling that $|\sin(\xi)|\leqslant|\xi|$ we have

$$\begin{aligned}|f(\rho\cos\theta\sin\varphi,\rho\sin\theta\sin\varphi,\rho\cos\varphi)| &= \left|\frac{\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)}{\rho^2}\right| \\ &\leqslant \rho|\cos\theta||\sin\theta||\sin\varphi|^2|\cos\varphi| \\ &\leqslant \rho\longrightarrow 0,\end{aligned}$$

when $\rho\longrightarrow 0$. Therefore the limit exists and is 0. □

We can extend these ideas to the case

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})=\ell,\ \ \vec{x}_0\in\mathbb{R}^d,\ \ell\in\mathbb{R}.$$

We have the

Proposition 1.2.10

Let $f: D\subset\mathbb{R}^d\longrightarrow\mathbb{R}$, $\vec{x}_0\in \mathrm{Acc}(D)\cap\mathbb{R}^d$. Suppose that

i) $|f(\vec{x})-\ell|\leqslant\phi(\|\vec{x}-\vec{x}_0\|)$;

ii) $\lim_{\rho\to 0+}\phi(\rho)=0$.

Then $\exists\lim_{\vec{x}\to\vec{x}_0} f(\vec{x})=\ell$.

In the case of

$$\lim_{\vec{x}\to\infty_d} f(\vec{x})=\ell\in\mathbb{R},$$

previous argument can be adapted in the following way: if

i) $|f(\vec{x})-\ell|\leqslant\phi(\|x\|)$;

ii) $\lim_{\rho\to+\infty}\phi(\rho)=0$.

Then $\lim_{\vec{x}\to\infty_d} f(\vec{x})=\ell$. This strategy can be used also to prove infinite limits, that limits whose value is infinity.

Example 1.2.11. *Compute*

$$\lim_{(x,y)\to\infty_2}\left(x^4+y^4-xy\right).$$

Sol. — Looking at the sections along the axes we have $f(x,0)=x^4\longrightarrow+\infty$ and $f(0,y)=y^4\longrightarrow+\infty$. So, if the limit exists must be $+\infty$. This seems reasonable because x^4+y^4 should dominate xy. Let us write f in polar coordinates:

$$f(\rho\cos\theta,\rho\sin\theta)=\rho^4(\cos\theta)^4+\rho^4(\sin\theta)^4-\rho^2(\cos\theta)(\sin\theta)=\rho^4\left[(\cos\theta)^4+(\sin\theta)^4\right]-\frac{1}{2}\rho^2\sin(2\theta).$$

Now: notice that the quantity $K(\theta):=(\cos\theta)^4+(\sin\theta)^4$ is always positive and has a minimum as $\theta\in[0,2\pi]$. Indeed: we don't need any computation because K is clearly continuous, hence K has a minimum by Weierstrass's theorem. Moreover $K(\theta) = 0$ iff $\cos\theta = \sin\theta = 0$, and this in impossible. We call C the minimum value of K: $K(\theta) \geqslant C > 0$ for any $\theta \in [0, 2\pi]$. Recalling also that $|\sin(2\theta)| \leqslant 1$ we have

$$f(x, y) \geqslant C\rho^4 - \frac{1}{2}\rho^2 \sin(2\theta) \geqslant C\rho^4 - \frac{1}{2}\rho^2 =: \phi(\rho) \longrightarrow +\infty.$$

By this the conclusion follows. □

In previous example, we used the following argument:

i) $f(\vec{x}) \geqslant \phi(\|\vec{x}\|)$,

ii) $\lim_{\rho\to+\infty} \phi(\rho) = +\infty$.

Then $\exists \lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty$. Here below more examples.

Example 1.2.12. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2 + y^2 + z^2)^2 - xyz\right].$$

Sol. — A quick check on the sections along the axes show that they tend to $+\infty$. Again: it seems reasonable that the fourth order term $(x^2 + y^2 + z^2)^2$ dominates on xyz. Passing to spherical coordinates

$$f = (\rho^2)^2 - \rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi) = \rho^4 - \frac{1}{4}\rho^3(\sin(2\theta))(\sin(2\varphi))(\sin\varphi).$$

Now, because

$$|(\sin(2\theta))(\sin(2\varphi))(\sin\varphi)| \leqslant 1,$$

we have

$$f \geqslant \rho^4 - \frac{1}{4}\rho^3 =: \phi(\rho) \longrightarrow +\infty,$$

from which the conclusion follows. □

Example 1.2.13. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2 + y^2)^2 + z^2 - xy\right]$$

Sol. — Easily the sections are all convergent to $+\infty$ (e.g. $f(x, 0, 0) = x^4 \longrightarrow +\infty$ when $\|(x, 0, 0)\| = |x| \longrightarrow +\infty$). In this case it is convenient to introduce *cylindrical coordinates*

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases}$$

because $x^2 + y^2 = \rho^2$. But be careful: $(x, y, z) \longrightarrow \infty_3$ means $\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} \longrightarrow +\infty$, and this doesn't mean necessarily that $\rho \longrightarrow +\infty$. However,

$$f_{cil} = (\rho^2)^2 + z^2 - \rho^2\cos\theta\sin\theta \geqslant \rho^4 + z^2 - \rho^2, \quad (|\cos\theta\sin\theta| \leqslant 1).$$

Now: if we had $f(x, y, z) \geqslant \rho^2 + z^2 = \|(x, y, z)\|^2$ we would be done. To this aim we may hope that $\rho^4 - \rho^2 \geqslant \rho^2$ and indeed this is actually true if ρ is big enough but not for every ρ. To get a lower bound true for any ρ we may notice that

$$\exists K \;:\; \rho^4 - \rho^2 \geqslant \rho^2 + K, \; \forall\rho.$$

Indeed: this is equivalent to say that $\rho^4 - 2\rho^2 \geqslant K$, that is the function $\rho \longmapsto \rho^4 - 2\rho^2$ is bounded below. But a quick check shows that this function has a global minimum: so, if we call K the minimum of the function $\rho \longmapsto \rho^4 - 2\rho^2$ we have the conclusion. □

1.3. Continuity

One of the major application of the concept of limit is to the definition of continuity:

Definition 1.3.1

Let $\vec{F} = \vec{F}(\vec{x}) \ : \ D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0 \in D \cap \mathrm{Acc}(D)$. We say that $\vec{F}$ **is continuous at** $\vec{x}_0$ if

$$\lim_{\vec{x} \to \vec{x}_0} \vec{F}(\vec{x}) = \vec{F}(\vec{x}_0).$$

If $\vec{F}$ is continuous in any point of D we say that $\vec{F}$ **is continuous on** D and we write $\vec{F} \in \mathscr{C}(D)$.

Sum, difference of continuous functions are continuous. Product makes sense if one of the two function is numerical and the other is either numerical or vector valued. In this case, product of continuous functions (provided well defined) is continuous. And, similarly, ratio of continuous functions is continuous. Also the composition of continuous functions is continuous.

It is easy to prove that, for vector valued functions, continuity of $\vec{F} = (f_1, \dots, f_m)$ holds iff each of its components is continuous:

Proposition 1.3.2

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0 \in D \cap \mathrm{Acc}(D)$. Then

$$\vec{F} = (f_1, \dots, f_m) \text{ is continuous at } \vec{x}_0 \iff f_j \text{ is continuous at } \vec{x}_0,\ \forall j = 1, \dots, m.$$

Let us focus on numerical functions $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. It is clear that every monomial

$$a x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \in \mathscr{C}(\mathbb{R}^d).$$

Thus, any polynomial, that is any finite sum of monomials is $\mathscr{C}(\mathbb{R}^d)$. Since composition of continuous functions is continuous, it is easy to draw that *any elementary function of a polynomial is continuous where defined.*

Example 1.3.3. *Where is continuous the function $f(x, y) := \log(1 - x^2 - y^2)$?*

Sol. — The function is defined on

$$D = \left\{(x, y) \in \mathbb{R}^2 \ : \ 1 - x^2 - y^2 > 0\right\} = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\right\}.$$

On D, f is log of a polynomial $1 - x^2 - y^2$, thus it is continuous on its domain. □

Example 1.3.4. *Euclidean norm is continuous on $\mathbb{R}^d$.*

Sol. — Just remind that

$$\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_d^2},$$

that is, $\|\vec{x}\|$ is root of a polynomial, then it is continuous where defined. Since the norm makes sense for every vector $\vec{x}$ we conclude that $\|\vec{x}\| \in \mathscr{C}(\mathbb{R}^d)$. □

Example 1.3.5 (Important). *Let A be an $m \times d$ matrix and define*

$$\vec{F}(\vec{x}) := Ax \equiv \begin{bmatrix} a_{11} & \dots & a_{1d} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{md} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \equiv \begin{pmatrix} a_{11}x_1 + \ldots + a_{1d}x_d \\ \vdots \\ a_{m1}x_1 + \ldots + a_{md}x_d \end{pmatrix},$$

$\vec{F}$ is called **linear map** *and we write $\vec{F} \in \mathscr{L}(\mathbb{R}^d, \mathbb{R}^m)$ (set of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^m$). We have $\vec{F} \in \mathscr{C}(\mathbb{R}^d)$.*

Sol. — Since

$$\vec{F} = (f_1, \ldots, f_m), \text{ where } f_j(x_1, \ldots, x_d) = a_{j1}x_1 + \cdots + a_{jd}x_d \in \mathscr{C}(\mathbb{R}^d), \ j = 1, \ldots, m,$$

we conclude that $\vec{F} \in \mathscr{C}(\mathbb{R}^d)$. □

1.4. Basic Topologial Concepts

Intervals play a special role with functions of real variable real valued, $f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. They are natural sets and they enter in most of the properties of continuity, differentiability and integrability. Certain specific properties of intervals are implicitly used to prove important results on functions. These properties are sort of qualitative properties as the fact that *any interval is made by one single piece*, or *an interval of type $[a, b]$ is bounded and it contains the endpoints*, or, again, *any point of an interval of type $]a, b[$ is in the interior of the interval itself.*

When we move to the case of vector valued functions of vector variable, $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, domain D is a subset of $\mathbb{R}^d$ and in $\mathbb{R}^d$ there is not a "natural set" as an interval is for $\mathbb{R}$. We need then to introduce a number of new concepts that are implicitly verified by intervals. These concepts involve properties of points respect to a given set and they go under the name of *Tolopogy*, which literally means *stufy of locations*. The first key concept is the following

Definition 1.4.1

Let $\vec{x}_0 \in \mathbb{R}^d$ and $r > 0$. We call **closed ball centred at $\vec{x}_0$ with radius** r the set

$$B(\vec{x}_0, r] := \left\{ \vec{x} \in \mathbb{R}^d \ : \ \|\vec{x} - \vec{x}_0\| \leqslant r \right\}.$$

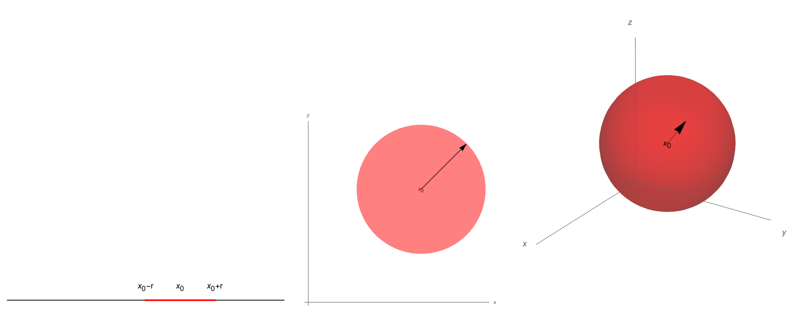


Figure 2. Balls in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

Of course, it is hard to visualize a ball of dimension $d \geqslant 4$, but it still makes perfect sense. Balls can be used to review the concept of *accumulation point*:

Proposition 1.4.2

Let $D \subset \mathbb{R}^d$. Then
- $\vec{x}_0 \in \mathbb{R}^d$ is accumulation point for D iff $(B(\vec{x}_0, r]\backslash\{\vec{x}_0\}) \cap D \neq \varnothing, \forall r > 0$;
- ∞_d is an accumulation point for D iff $B(\vec{0}, r]^c \cap D \neq \varnothing, \forall r > 0$.

The proof of the previous Proposition is an important exercise left to the reader.

Definition 1.4.3

Let $D \subset \mathbb{R}^d$. A point $\vec{x}_0 \in \mathbb{R}^d$ is said to be
- in the **interior of** D if
$$\exists\, B(\vec{x}_0, r] \subset D.$$
(thus, in particular, $\vec{x}_0$ itself belongs to D). Set of interior points of D is denoted by $\mathrm{Int}(D)$ and it is called **interior of** D.
- in the **boundary of** D if every ball centred at $\vec{x}_0$ contains at same time points of D and points of D^c, that is
$$B(\vec{x}_0, r] \cap D \neq \varnothing, \quad B(\vec{x}_0, r] \cap D^c \neq \varnothing, \quad \forall r > 0.$$
The boundary of D is denoted with ∂D.

Let us see some simple examples. We will not provide detailed justifications in all cases (it might be hard!).

- In $\mathbb{R}$, $\mathrm{Int}([a, b]) =]a, b[$, $\partial[a, b] = \{a, b\}$. Here $D = [a, b]$. Recalling that $B(x_0, r] = [x_0 - r, x_0 + r]$, it is easy to check that every point $x_0 \in]a, b[$ is contained in $D = [a, b]$ with a suitable interval $[x_0 - r, x_0 + r]$. This is not true for endpoints a, b because, for example $[a-r, a+r]$ contains the subinterval $[a-r, 0[$ which has no points in D, thus $[a-r, a+r] \not\subset D$. About boundary: at a, every ball $[a-r, a+r]$ has point of D and of D^c (recall here $D = [a, b]$) and same for b. If $x_0 \in]a, b[$ and the ball $[x_0 - r, x_0 + r] \subset [a, b]$ then there cannot be points of D^c in such a ball. Same for points out of $[a, b]$. This explains $\partial[a, b] = \{a, b\}$.
- In $\mathbb{R}^d$,
$$\begin{aligned} &\mathrm{Int} B(\vec{x}_0, r] = \left\{\vec{x} \in \mathbb{R}^d \;:\; \|\vec{x} - \vec{x}_0\| < r\right\} =: B(\vec{x}_0, r[, \\ &\partial B(\vec{x}_0, r] = \left\{\vec{x} \in \mathbb{R}^d \;:\; \|\vec{x} - \vec{x}_0\| = r\right\}. \end{aligned}$$
Intuitively clear, some work has to be done to prove details (see exercises).

Definition 1.4.4

A set $D \subset \mathbb{R}^d$ is said to be **open** if $\mathrm{Int}(D) = D$, that is if every point of D lies in D with an entire ball. Empty set $\varnothing$ is considered open by definition.

So, for example, $B(\vec{x}_0, r[$ (open ball) is open (in the sense of the previous definition) but $B(\vec{x}_0, r]$ is not open (because points of the *edge* $\{\vec{x} \ : \ \|\vec{x} - \vec{x}_0\| = r\}$ are not in the interior of $B(\vec{x}_0, r])$.

Definition 1.4.5

A set $D \subset \mathbb{R}^d$ is said to be **closed** if D^c, is open.

Clearly $\mathbb{R}^d$ is open (obvious) and since $(\mathbb{R}^d)^c = \emptyset$ is open, $\mathbb{R}^d$ is also closed. Similarly, $\emptyset$ is (by definition) open, and since $\emptyset^c = \mathbb{R}^d$ is open, $\emptyset$ is also closed. This shows that a set could be both open and closed. This remark is important because someone may think that a set D is either open or closed. This is wrong! As we said, there are sets both open and closed. There are also sets which are neither open nor closed. For example, in $\mathbb{R}$, interval $[a, b[$ is not open (because $a \notin \mathrm{Int}([a, b]))$ nor closed (because $[a, b[^c =]-\infty, a[\cup [b, +\infty[$ is not open since b does not lie in its interior). For your information, $\mathbb{R}^d$ and $\emptyset$ are the unique subsets of $\mathbb{R}^d$ both open and closed.

An important characterization of closed sets is provided by the following

Proposition 1.4.6: Cantor

A set $D \subset \mathbb{R}^d$ is closed if and only if it contains all possible finite limit of convergent sequences in D. Formally:

$$D \text{ closed} \iff \forall (\vec{x}_n) \subset D \ : \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \text{ then } \vec{\ell} \in D.$$

Proof. $\Longrightarrow$ Assume D closed and let $(\vec{x}_n) \subset D$ be such that $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. The claim is: $\vec{\ell} \in D$. If false, $\vec{\ell} \in D^c$, since D^c is open (by assumption, D is closed), then

$$\exists\, B(\vec{\ell}, r] \subset D^c.$$

Since $\vec{x}_n \longrightarrow \vec{\ell}$, according to the definition of limit,

$$\exists N \ : \ \|\vec{x}_n - \vec{\ell}\| \leqslant r, \ \forall n \geqslant N, \ \Longrightarrow \ \vec{x}_n \in B(\vec{\ell}, r] \subset D^c, \ \forall n \geqslant N,$$

and this contradicts $(\vec{x}_n) \subset D$.

$\Longleftarrow$ Assume that D contains all possible finite limit of convergent sequences in D. The goal is to prove that D is closed, that is D^c is open. If $D^c = \emptyset$ there is nothing to prove. So assume $D^c \neq \emptyset$ and pick a point $\vec{\ell} \in D^c$. We have to prove that

$$\exists\, B(\vec{\ell}, r] \subset D^c.$$

Suppose, by contradiction, that ball does not exist. Then,

$$\forall r > 0, \ B(\vec{\ell}, r] \cap D \neq \emptyset.$$

Take $r = \frac{1}{n}$: we have

$$\forall n \in \mathbb{N} \backslash \{0\}, \ \exists \vec{x}_n \in D \ : \ \|\vec{x}_n - \vec{\ell}\| \leqslant \frac{1}{n}.$$

Then, $(\vec{x}_n) \subset D$ and $\vec{x}_n \longrightarrow \vec{\ell}$. By assumption, necessarily $\vec{\ell} \in D$ and this contradicts $\vec{\ell} \in D^c$. $\square$

Open and closed sets are important classes of sets as we will see. It is therefore important to have general and easily testable conditions to ensure whether a set D is open or closed. A common way to define sets in $\mathbb{R}^d$ is through a certain number of equations or inequalities (strict or large).

FIGURE 3. The set $D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \leqslant 1, \ \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\right\}$

It turns out that continuity is an important key to easily showing the topological nature of a set:

Proposition 1.4.7

Any set defined through a finite number of strict inequalities involving continuous functions is open. Any set defined through a finite number of large inequalities and/or equalities involving continuous functions is closed. Formally, let $g_1, \dots, g_m, h_1, \dots, h_k \in \mathscr{C}(\mathbb{R}^d)$. Then

$$D := \left\{\vec{x} \in \mathbb{R}^d \ : \ g_1(\vec{x}) < 0, \dots, g_m(\vec{x}) < 0\right\} \text{ is } \textbf{open},$$

$$D := \left\{\vec{x} \in \mathbb{R}^d \ : \ g_1(\vec{x}) \leqslant 0, \dots, g_m(\vec{x}) \leqslant 0, \ h_1(\vec{x}) = 0, \dots, h_k(\vec{x}) = 0\right\} \text{ is } \textbf{closed}.$$

PROOF. For simplicity, consider $D = \{\vec{x} \in \mathbb{R}^d \ : \ g(\vec{x}) < 0\}$. Since $D = \{\vec{x} \in \mathbb{R}^d \ : \ g(\vec{x}) \geqslant 0\}^c =: C^c$, if we prove that C is closed, by definition $D = C^c$ must be open. If $C = \varnothing$ there is nothing to prove. Otherwise we use the Cantor's characterization to show that C is closed. That is, we have to prove that

$$\text{if } (\vec{x}_n) \subset C, \ : \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \implies \vec{\ell} \in C.$$

We know

$$\vec{x}_n \in C, \implies g(\vec{x}_n) \geqslant 0.$$

Since g is continuous and $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$, then $g(\vec{x}_n) \longrightarrow g(\vec{\ell})$. According to the permanence of sign, $g(\vec{\ell}) \geqslant 0$, that is $\vec{\ell} \in C$ as advertised. This shows that C is closed, hence D is open.

To prove that $D := \{\vec{x} \in \mathbb{R}^d \ : \ g(\vec{x}) \leqslant 0\}$ is closed follows basically by the same argument we have shown here above (the unique difference is the $\geqslant$ that becomes a $\leqslant$). $\square$

1.5. Weierstrass' Theorem

The search for minimum/maximum points of a numerical function is one of the most important problems in many applications. We consider here the case of a numerical function of vector variable:

Definition 1.5.1

Let $f = f(\vec{x}) \;:\; D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We say that $\vec{x}_{min} \in D$ is a **global minimum point for** f **on** D if

$$f(\vec{x}_{min}) \leqslant f(\vec{x}),\ \forall \vec{x} \in D.$$

We call **minimum value of** f **on** D the value of f at minimum point $\vec{x}_{min}$, that is $f(\vec{x}_{min})$. We write

$$\min_D f \equiv \min_{\vec{x} \in D} f(\vec{x}) := f(\vec{x}_{min}).$$

Similar definitions for **maximum point for** f **on** D and **maximum value of** f **on** D (denoted by $\max_D f$ or $\max_{\vec{x}\in D} f(\vec{x})$).

We recall that every $f \in \mathscr{C}([a,b])$ has global minimum/maximum over $[a,b]$. The conclusion is false if the interval $[a,b]$ is not closed and bounded. This is the important Weierstrass' theorem. We look at an extension of this result to the case of functions of vector variable $f = f(\vec{x}) \;:\; D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We may expect that, under suitable assumptions on D, if $f \in \mathscr{C}(D)$ then f will have global min/max on D. The right general conditions on D are that this must be **closed** and **bounded**:

Definition 1.5.2

A set $D \subset \mathbb{R}^d$ is **bounded** if

$$\exists M \;:\; \|\vec{x}\| \leqslant M,\ \forall \vec{x} \in D.$$

We have now all the ingredients for the

Theorem 1.5.3: Weierstrass

Every continuous function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ on a closed and bounded domain D has global minimum and global maximum on D.

Weierstrass' theorem points out the importance of the class of closed and bounded subsets of $\mathbb{R}^d$:

Definition 1.5.4

A set $D \subset \mathbb{R}^d$ is **compact** if it is closed and bounded.

Example 1.5.5. *Every closed ball $B(\vec{x}_0, r]$ is compact.*

Sol. — First, the ball is closed because defined by a large inequality involving a continuous function. Indeed

$$\vec{x} = (x_1, \dots, x_d) \in B(\vec{x}_0, r], \iff \sqrt{(x_1 - x_{1,0})^2 + \cdots + (x_n - x_{n,0})^2} \leqslant r.$$

Second, the ball is trivially bounded: since $\|\vec{x} - \vec{x}_0\| \leqslant r$, we have

$$\|\vec{x}\| = \|\vec{x} - \vec{x}_0 + \vec{x}_0\| \overset{\triangle}{\leqslant} \|\vec{x} - \vec{x}_0\| + \|\vec{x}_0\| \leqslant r + \|\vec{x}_0\| =: M,\ \forall \vec{x} \in B(\vec{x}_0, r]. \quad \square$$

If it is relatively easy to check if a set is closed, less trivial is to show that is it bounded.

Example 1.5.6. *Show that the set* $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1, \ y^2 + z^2 \leqslant 1\}$ *is compact.*

Sol. — D is closed being defined through large inequalities or equalities involving continuous functions. Let us see it is also bounded. Notice that, if $(x, y, z) \in D$, $y^2 + z^2 \leqslant 1$, thus $y^2 \leqslant 1$ and $z^2 \leqslant 1$. By the first relation,

$$x^2 + y^2 = 1 + z^2, \implies x^2 = 1 + z^2 - y^2 \leqslant 1 + z^2 \leqslant 1 + 1 = 2,$$

by which $x^2 \leqslant 2$. Therefore

$$\|(x, y, z)\|^2 = x^2 + y^2 + z^2 \leqslant 2 + 1 + 1 = 4 =: M, \ \forall (x, y, z) \in D.$$

This proves that D is also bounded, hence compact. □

Example 1.5.7. *Discuss whether* $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x = yz + 1\}$ *is compact or less.*

Sol. — Clearly, D is defined through an equation involving a continuous function, it is therefore closed. About boundedness, notice that D contains points of type $(y^2 + 1, y, y)$ for every $y \in \mathbb{R}$. Now since

$$\|(y^2 + 1, y, y)\|^2 = (y^2 + 1)^2 + 2y^2 \longrightarrow +\infty, \ y \longrightarrow \pm\infty,$$

we conclude that there cannot be a constant M such that $\|(x, y, z)\| \leqslant M$ for every $(x, y, z) \in D$. Therefore, D is not bounded, hence certainly not compact. □

When the domain D on which f is continuous is closed and unbounded, Weierstrass' theorem does not apply. In this case, in certain situations we might still ensure existence of min/max by adding some further assumption. The following result is actually a consequence of Weierstrass' theorem

Proposition 1.5.8

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on D, closed and unbounded, such that

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty \ (-\infty).$$

Then f has a global minimum (maximum).

Proof. We do the proof for the minimum, the other case being similar. Assume then that

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty.$$

Pick a point $\vec{x}_0 \in D$ and consider the new domain

$$\widetilde{D} := \{\vec{x} \in D \ : \ f(\vec{x}) \leqslant f(\vec{x}_0) + 1\}.$$

Notice that, in particular, $\vec{x}_0 \in \widetilde{D}$. Clearly $\widetilde{D} \subset D$ and since $\widetilde{D}$ is defined through an inequality involving a continuous function, it is closed. We claim that $\widetilde{D}$ is also bounded. If false,

$$\forall n \in \mathbb{N}, \ \exists \vec{x}_n \in \widetilde{D} \subset D, : \ \|\vec{x}_n\| \geqslant n.$$

Then $\vec{x}_n \longrightarrow \infty_d$, thus $f(\vec{x}_n) \longrightarrow +\infty$, and this is impossible since $f(\vec{x}_n) \leqslant f(\vec{x}_0) + 1$ because $\vec{x}_n \in \widetilde{D}$.

Since $\widetilde{D}$ is closed and bounded, that is compact, Weierstrass' theorem applies to f on $\widetilde{D}$: there exists a point $\vec{x}_{min} \in \widetilde{D} \subset D$ such that

$$f(\vec{x}_{min}) \leqslant f(\vec{x}), \ \forall \vec{x} \in \widetilde{D}.$$

If now $\vec{x} \in D \backslash \widetilde{D}$ it means that $f(\vec{x}) \geqslant f(\vec{x}_0) + 1 > f(\vec{x}_0) \geqslant f(\vec{x}_{min})$. We conclude that, no matter where is taken $\vec{x} \in D$, $f(\vec{x}) \geqslant f(\vec{x}_{min})$, that is $\vec{x}_{min}$ is a global minimum for f. □

Example 1.5.9. *Show that the function $f(x,y) := x^4 + y^4 - xy$ has global minimum on $\mathbb{R}^2$. What about global maximum?*

Sol. — Of course $f \in \mathscr{C}(\mathbb{R}^2)$ (because it is a polynomial). Notice that $\mathbb{R}^2$ is closed but unbounded. We have also seen (see Example 1.2.11) that

$$\lim_{(x,y)\to\infty_2} f(x,y) = +\infty.$$

Therefore, by the Corollary of Weierstrass's thm we have that there exists a global minimum for f on $\mathbb{R}^2$. On the other side, because f is upper unbounded (by the limit at ∞_2) the global maximum doesn't exists. □

Weiestrass' theorem is a pure *existence* result, it does not provide any concrete method to find global min/max points. This will be provided by Differential Calculus we will develop in the next Chapter.

1.6. Intermediate values theorem

Another important property of continuous functions of one real variable is that, on intervals, if a function takes both positive and negative values, then it must take also value 0 (zeroes theorem). More in general, if the function takes any two numbers, the it takes all values between these two numbers (intermediate values theorem). Is this somehow still true if $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is function of vector variable? Under which assumptions on D?

The remarkable property fulfilled by intervals (independently from their specific shape) is that they are "made by one single piece". We will now introduce a definition to formulate this concept for a set $D \subset \mathbb{R}^d$:

Definition 1.6.1

We say that D is **connected** if any two points of D are joint by a continuous curve in D, that is

$$\forall \vec{x}, \vec{y} \in D,\ \exists\, \vec{\gamma} = \vec{\gamma}(t) : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d,\ :\ \vec{\gamma} \subset D,\ \vec{\gamma} \in \mathscr{C}([a,b]),\ :\ \vec{\gamma}(a) = \vec{x},\ \vec{\gamma}(b) = \vec{y}.$$

To check whether a set D is connected or not it might be difficult. We will content here with a practical intuition.

Theorem 1.6.2

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ continue on D connected. Then, if f takes any two values, it takes also all other values between these two.

Proof. Let $\xi, \eta \in f(D)$, that is $\xi = f(\vec{x})$ and $\eta = f(\vec{y})$ for some $\vec{x}, \vec{y} \in D$. Since D is connected, there exists a continuous curve $\vec{\gamma} \subset D$ such that $\vec{\gamma}(a) = \vec{x}$ and $\vec{\gamma}(b) = \vec{y}$. Define then

$$g : [a,b] \longrightarrow \mathbb{R},\ g(t) := f(\vec{\gamma}(t)).$$

Since g is composition of continuous functions it is itself continuous. Moreover $g(a) = \xi$ and $g(b) = \eta$. According to the ordinary intermediate values theorem, g takes all values between ξ and η. But this means that

$$\forall \zeta \in [\xi, \eta],\ \exists t \in [a,b]\ :\ \zeta = g(t) = f(\vec{\gamma}(t)),$$

that is $[\xi, \eta] \subset f(D)$ and this is the conclusion. □

Notice that if $f \in \mathscr{C}(D)$ where D is compact and connected, then

$$f(D) = [\min_D f, \max_D f].$$

This because $\min_D f$ and $\max_D f$ are assumed at $\vec{x}_{min}$ and $\vec{x}_{max}$ respectively. Thus $[\min_D f, \max_D f] \subset f(D)$. And because $\min_D f$ and $\max_D f$ are, resp., the minimum and maximum values of f on D, $f(D) = [\min_D f, \max_D f]$.

1.7. Exercises

Exercise 1.7.1. By using the definition of norm, prove the *parallelogram identity*:

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2, \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d.$$

Exercise 1.7.2. For each of the following sequences of vectors say whether it is convergent and, if yes, what is the limit for $n \longrightarrow +\infty$:

i) $\vec{x}_n := (e^{-n}, 1)$.
ii) $\vec{x}_n := \left(n, n^2\right)$.
iii) $\vec{x}_n := \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right)$.
iv) $\vec{x}_n := \left(1, 1+\frac{1}{n}, n\right)$.
v) $\vec{x}_n := \left(\tanh n, \frac{\log n}{n}, \frac{\sin n}{n}\right)$.
vi) $\vec{x}_n := \left((-1)^n, (-1)^{n+1}\right)$.

Exercise 1.7.3 ($\star$). For each of the following statements provide a proof (if true) or a counterexample (if false):

- if $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \vec{0}$ then $x_{n,j} \longrightarrow 0$ for every $j = 1, \dots, d$.
- If $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$ then $|x_{n,j}| \longrightarrow +\infty$, $j = 1, \dots, d$.
- if $x_{n,j} \longrightarrow +\infty$ for at least one component j then $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$.
- if $\vec{x}_n = (x_{n,1}, \dots, x_{n,d})$ does not have limit, then all components $x_{n,j}$ ($j = 1, \dots, d$) do not have limit as well.

Exercise 1.7.4. Write parametric form $\vec{\gamma} = \vec{\gamma}(t)$ for each of the following plane curves:

i) $3x + 2y = 1$.
ii) $x = 5$.
iii) $y = x^2$.
iv) $x = y^3$.
v) $x^2 + y^2 = 1$.
vi) $x^2 + 2y^2 = 3$.

Exercise 1.7.5. Let $f(x, y) := \frac{xy^2}{x^2+y^4}$ on $D = \mathbb{R}^2 \backslash \{\vec{0}\}$. Compute limits of f along the following sections:

i) $\vec{\gamma}(t) = (t, 0)$, $t \longrightarrow 0$.
ii) $\vec{\gamma}(t) = (0, t)$, $t \longrightarrow 0$.
iii) $\vec{\gamma}(t) = e^{-t}(\cos t, \sin t)$, $t \longrightarrow +\infty$.

What can you draw about $\lim_{(x,y)\to\vec{0}} f(x, y)$?

Exercise 1.7.6. Looking at suitable sections, prove that the following limits do not exist:

1. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 - y^2}{x^2 + y^2}$.
2. $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 + y^3}{x^2 + y^2}$.
3. $\displaystyle\lim_{(x,y)\to 0_2} \frac{y^2 - xy}{x^2 + y^2}$.
4. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{x + y^2 + z^3}{\sqrt{x^2 + y^2 + z^2}}$.
5. $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy + \sqrt{y^2+1} - 1}{x^2 + y^2}$.
6. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{xz}{x^4 + y^2 + z^2}$.

Exercise 1.7.7. Compute the following limits:

1. $\lim_{(x,y)\to 0_2} \dfrac{xy}{\sqrt{x^2+y^2}}$. 2. $\lim_{(x,y)\to 0_2} \dfrac{x^2y^3}{(x^2+y^2)^2}$. 3. $\lim_{(x,y)\to 0_2} \dfrac{x^3-y^3}{x^2+y^2}$. 4. $\lim_{(x,y)\to 0_2} \dfrac{x\sqrt{|y|}}{\sqrt[3]{x^4+y^4}}$. 5. $\lim_{(x,y)\to 0_2} \dfrac{xy}{|x|+|y|}$.

Exercise 1.7.8. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\lim_{(x,y)\to 0_2} \dfrac{e^{4y^3}-\cos(x^2+y^2)}{x^2+y^2}$. 2. $\lim_{(x,y,z)\to 0_3} \dfrac{xyz}{x^2+y^2+z^2}$. 3. $\lim_{(x,y,z)\to 0_3} \dfrac{(x^2+yz)^2}{\sqrt{(x^2+y^2)^2+z^4}}$.

4. $\lim_{(x,y)\to 0_2} \dfrac{\log(1+2x^3)}{\sinh(x^2+y^2)}$. 5. $\lim_{(x,y)\to(0,1)} \dfrac{x^3\sinh(y-1)}{x^2+y^2-2y+1}$. 6. $\lim_{(x,y)\to(1,1)} \dfrac{(x-1)^2(y-1)^7}{\left((x-1)^2+(y-1)^2\right)^{5/2}}$.

Exercise 1.7.9. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\lim_{(x,y)\to\infty_2} \left(x^3+xy^2-y^2\right)$. 2. $\lim_{(x,y)\to\infty_2} (x^4-y^4+y^2-x^2)$.

3. $\lim_{(x,y)\to\infty_2} \left(x^2y^2+x^2+y^2-xy\right)$. 4. $\lim_{(x,y,z)\to\infty_3} \left(x^4+y^4+z^4-xyz\right)$.

5. $\lim_{(x,y,z)\to\infty_3} \left(x^2+y^2+z^4-xz\right)$. 6. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{x^2+y^2}+z^2-z\right)$.

7. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{(x^2+y^2)^2+z^4}-xyz\right)$.

Exercise 1.7.10. Prove that

- $\mathrm{Int}B(\vec{x}_0,r] = \{\vec{x} \ : \ \|\vec{x}-\vec{x}_0\| < r\}$.
- $\partial B(\vec{x}_0,r] = \{\vec{x} \ : \ \|\vec{x}-\vec{x}_0\| = r\}$.

Exercise 1.7.11 (★). Prove Proposition 1.4. (it is a double implication).

Exercise 1.7.12. For each of the following sets say if it is open, closed, bounded, compact.

i) $D := \{(x,y)\in\mathbb{R}^2 \ : \ xy>0\}$.
ii) $D := \{(x,y)\in\mathbb{R}^2 \ : \ |xy|\leqslant 1\}$.
iii) $D := \{(x,y)\in\mathbb{R}^2 \ : \ |x|+|y|<1\}$.
iv) $D := \{(x,y)\in\mathbb{R}^2 \ : \ 1\leqslant xy\leqslant 2,\ x\leqslant y\leqslant 2x\}$.
v) $D := \{(x,y,z)\in\mathbb{R}^3 \ : \ x^2+2y^2+3z^4<4\}$.
vi) $D := \{(x,y,z)\in\mathbb{R}^3 \ : \ z\geqslant x^2+y^2,\ x^2+z^2\leqslant 1\}$.
vii) $D := \{(x,y,z)\in\mathbb{R}^3 \ : \ x^2-y^2+z^2\leqslant 1\}$.
viii) $D := \{(x,y,z)\in\mathbb{R}^3 \ : xy=z+1,\ x^2+y^2\leqslant 1\}$.

CHAPTER 2

Differential Calculus

In this chapter we extend the *Differential Calculus* to the case of *vector-valued functions of several variables*,

$$\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m.$$

The extension to the multi dimensional case of the concept of derivative is not straightforward. Indeed, we cannot just write

$$\vec{F}'(\vec{x}) := \lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}{\vec{h}}$$

because the division by a vector is not defined. This difficulty is due to the nature of the domain $\mathbb{R}^d$. We will see two possible ways to solve this issue. First, we will introduce the concept of *directional derivative* (which only in part solves the issue). Hence, we will introduce the true solution: the concept of *differential*. Differential Calculus has a large number of applications. In this Chapter we will illustrate optimization techniques to solve unconstrained and constrained optimization problems.

Chapter requirements: a good comprehension of ordinary derivative, basic linear algebra (vector spaces, algebra of matrices).

2.1. Directional derivative

The first way to bypass the technical difficulty of dividing by $\vec{h}$ is to assume that $\vec{h} = t\vec{v}$ where $t \in \mathbb{R}$ is variable and $\vec{v} \in \mathbb{R}^d$ is fixed. This leads to the

Definition 2.1.1

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x} \in \mathrm{Int}(D)$. We call **directional derivative of $\vec{F}$ at point $\vec{x}$ along $\vec{v} \neq \vec{0}$** the limit

$$D_{\vec{v}}\vec{F}(\vec{x}) := \lim_{t\to 0} \frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} \in \mathbb{R}^m.$$

Example 2.1.2. *Compute $D_{(1,1)}f(0,0)$ for $f(x,y) = x\cos y$.*

Sol. — We have

$$D_{(1,1)}f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{t\cos t}{t} = \lim_{t\to 0} \cos t = 1. \quad \square$$

Directional derivative does not provide a satisfactory definition of derivative. This because it may happens that all the $D_{\vec{v}}\vec{F}(\vec{x})$ exists but $\vec{F}$ is not even continuous!

EXAMPLE 2.1.3. *Let*

$$f(x,y) := \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & (x,y) \neq \vec{0}, \\ \\ 0, & (x,y) = \vec{0}. \end{cases}$$

Then f has all the directional derivatives at $\vec{0}$ but it is not therein continuous.

SOL. — Let us start with the continuity. Looking at principal sections. we have $f(x,0) = f(0,y) \equiv 0 \longrightarrow 0$. However, along the section $y = x^2$ we have

$$f(x,x^2) = \frac{x^2x^2}{x^4+x^4} = \frac{1}{2} \longrightarrow \frac{1}{2} \neq f(0,0) = 0.$$

Therefore $\nexists \lim_{(x,y)\to\vec{0}} f(x,y)$ and consequently the function cannot be continuous! Let us prove now that $\exists D_{\vec{v}} f(0,0)$ for any $\vec{v}$. Let $\vec{v} = (a,b) \neq \vec{0}$. We have

$$D_{\vec{v}} f(0,0) = \lim_{t\to 0} \frac{f\left((0,0)+t(a,b)\right) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(ta,tb)}{t} = \lim_{t\to 0} \frac{\frac{t^3a^2b}{t^2(t^2a^4+b^2)}}{t} = \lim_{t\to 0} \frac{a^2b}{t^2a^4+b^2},$$

that is

$$D_{\vec{v}} f(0,0) = \begin{cases} 0, & \text{if } b = 0 \text{ (and of course } a \neq 0), \\ \\ \frac{a^2}{b^2}, & \text{if } b \neq 0. \end{cases} \qquad \square$$

What is disturbing here is that we associate differentiability with *smoothness*. Even if directional derivative is not the right definition for a derivative, some special directional derivatives have great relevance:

Definition 2.1.4

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $x \in \mathrm{Int}(D)$ and let $\vec{e}_1, \ldots, \vec{e}_d$ be the canonical base of $\mathbb{R}^d$, that is $e_j = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the $j-$th place. We call **partial derivative of f with respect to the $j-$th variable at point** $\vec{x}$ the

$$\partial_j f(\vec{x}) := D_{\vec{e}_j} f(\vec{x}).$$

Partial derivative ∂_j is nothing but an ordinary derivatives w.r.t x_j considering all other variables x_i $i \neq j$ as fixed parameters. Indeed

$$\begin{aligned} \partial_j f(x) \quad &= \lim_{t\to 0} \frac{f\left((x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_d) + t(0,\ldots,0,1,0,\ldots,0)\right) - f(x_1,\ldots,x_d)}{t} \\ &= \lim_{t\to 0} \frac{f\left(x_1,\ldots,x_{j-1},x_j+t,x_{j+1},\ldots,x_d\right) - f(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_d)}{t} \end{aligned}$$

So, for instance

$$\partial_x\,(y\sin x) = y\cos x, \quad \partial_y\,(y\sin x) = \sin x.$$

2.2. Differential

Let
$$\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m.$$
We already noticed that we cannot use the one dimensional definition
$$\vec{F}'(\vec{x}) := \lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}{\vec{h}}$$
to define $\vec{F}'(\vec{x})$. For numerical functions, the ordinary definition of derivative can be rephrased in another equivalent form. Indeed,
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \iff \lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0,$$
that is
$$f(x+h) - f(x) - f'(x)h = o(h).$$
Here, $f'(x)$ is a number and $f'(x)h$ is the algebraic product between $f'(x)$ and h. Imagine now we wish to set this relation in general: we should write
$$\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h} = o(\vec{h}).$$
We now show that we can give a precise meaning to this relation. The first point is on the interpretation of $\vec{F}'(\vec{x})\vec{h}$. Since $\vec{F}(\vec{x}+\vec{h}), \vec{F}(\vec{x}) \in \mathbb{R}^m$, also $\vec{F}'(\vec{x})\vec{h} \in \mathbb{R}^m$. Moreover, reasonably $\vec{h} \longmapsto \vec{F}'(\vec{x})\vec{h}$ must be a *linear map* transforming $\vec{h} \in \mathbb{R}^d$ into $\vec{F}'(\vec{x})\vec{h} \in \mathbb{R}^m$. In other words
$$\vec{F}'(\vec{x}) \in \mathscr{L}(\mathbb{R}^d, \mathbb{R}^m).$$
Such type of transformation is represented by an $m \times d$ (lines$\times$ columns) matrix. Thus, we expect that $\vec{F}'(\vec{x})$ is a matrix. The second point involves the interpretation of $o(\vec{h})$. We cannot say that $g(\vec{h})$ is $o(\vec{h})$ if
$$\frac{g(\vec{h})}{\vec{h}} \longrightarrow \vec{0},$$
because of the same problem: we cannot divide per $\vec{h}$. However, the intuitive meaning of $o(\vec{h})$, a quantity smaller than $\vec{h}$, may lead to think that $g(\vec{h})$ is smaller order of $\vec{h}$ if it is smaller order of its length, that is $\|\vec{h}\|$. This leads to the following

Definition 2.2.1

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. We say that $\vec{F}$ is **differentiable** at point $\vec{x}$ if there exists an $m \times d$ matrix (denoted by $\vec{F}'(\vec{x})$) such that
$$\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h} = o(\vec{h}),$$
that is,

(2.2.1)
$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h}}{\|\vec{h}\|} = \vec{0}.$$

The matrix $\vec{F}'(\vec{x})$ is also called the **jacobian matrix** of $\vec{F}$ at the point $\vec{x}$.

The first question is: *how do we determine the entries of the Jacobian matrix?* Here we appreciate the concept of partial derivative:

Proposition 2.2.2

$$\exists \vec{F}'(\vec{x}), \implies \exists D_{\vec{v}}\vec{F}(\vec{x}) = \vec{F}'(\vec{x})\vec{v},\ \forall \vec{v} \in \mathbb{R}^d. \tag{2.2.2}$$

In particular, if $\vec{F} = (f_1, \dots, f_m)$ then

$$\vec{F}'(\vec{x}) = \begin{bmatrix} \partial_1 f_1(\vec{x}) & \partial_2 f_1(\vec{x}) & \dots & \partial_d f_1(\vec{x}) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_m(\vec{x}) & \partial_2 f_m(\vec{x}) & \dots & \partial_d f_m(\vec{x}) \end{bmatrix}. \tag{2.2.3}$$

Proof. Let us prove (2.2.2). Fix $\vec{v} \neq 0$. Then, since $\vec{F}$ is differentiable at $\vec{x}$,

$$\vec{F}(\vec{x}+t\vec{v}) - \left(\vec{F}(\vec{x}) + \vec{F}'(\vec{x})(tv)\right) = o(t\vec{v}), \implies \frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} = \vec{F}'(\vec{x})\vec{v} + \frac{o(t\vec{v})}{t}.$$

Now, since

$$\lim_{t\to 0}\left\| \frac{o(t\vec{v})}{t} \right\| = \lim_{t\to 0} \frac{\|o(t\vec{v})\|}{|t|} = \|\vec{v}\| \lim_{t\to 0} \frac{\|o(t\vec{v})\|}{\|t\vec{v}\|} = \|\vec{v}\| \lim_{\vec{h}\to\vec{0}} \frac{\|o(\vec{h})\|}{\|\vec{h}\|} = 0,$$

we conclude that

$$\lim_{t\to 0} \frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} = \vec{F}'(\vec{x})\vec{v}.$$

Let's prove now the (2.2.3): if we call $\vec{F}'(\vec{x}) = [a_{ij}]$, it is well known by Linear Algebra that

$$\vec{F}'(\vec{x})e_j$$

gives the $j-$th column of the matrix $\vec{F}'(\vec{x})$. So, the element a_{ij} of $\vec{F}'(\vec{x})$ is obtained by taking the $i-$th component of the vector $\vec{F}'(\vec{x})\vec{e}_j$. But: by (2.2.2) we have

$$\vec{F}'(\vec{x})\vec{e}_j = D_{\vec{e}_j}\vec{F}(\vec{x}) = \partial_j \vec{F}(\vec{x}) = \left(\partial_j f_1(\vec{x}), \partial_j f_2(\vec{x}), \dots, \partial_j f_m(\vec{x})\right),$$

hence the $i-$th component is $\partial_j f_i(\vec{x})$, and this proves (2.2.3). □

We mention a couple of important cases of the Jacobian matrix:

- $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$: in this case, $f'(\vec{x})$ is a $1 \times d$ matrix, precisely

$$f'(\vec{x}) = \left[\partial_1 f(\vec{x})\ \partial_2 f(\vec{x})\ \dots\ \partial_d f(\vec{x})\right] =: \nabla f(\vec{x}),$$

  is called **gradient of f at $\vec{x}$**. In this case

$$f'(\vec{x})\vec{h} = \nabla f(\vec{x}) \cdot \vec{h},$$

  where we denoted by $\cdot$ the scalar product of vectors in $\mathbb{R}^d$.
- $\vec{\gamma} : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d$: in this case $\vec{\gamma}'(t)$ is a $d \times 1$ matrix, precisely

$$\vec{\gamma}'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_d'(t) \end{bmatrix}.$$

Example 2.2.3. *Discuss the differentiability at* $(0,0)$ *of*

$$f(x,y) := \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Sol. — We know that the candidate for $f'(0,0,) = \nabla f(0,0)$ if the gradient exists. Notice that we cannot simply compute partial derivatives and evaluate in $(0,0)$ because, for instance,

$$\partial_x f(x,y) = \partial_x \frac{x^2y^2}{x^2+y^2} = \frac{2xy^2(x^2+y^2) - x^2y^2 2x}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2},$$

is of course not defined in $(0,0)$. In this case we have to proceed directly in the computation of $\partial_x f(0,0)$, that is

$$\partial_x f(0,0) = D_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0,$$

and similarly $\partial_y f(0,0) = 0$. We deduce $\nabla f(0,0) = (0,0)$. To prove that f is differentiable at $(0,0)$ we have to check that

$$f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0}) \cdot \vec{h} = o(\vec{h}), \iff \lim_{\vec{h}\to\vec{0}} \frac{\|f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0}) \cdot \vec{h}\|}{\|\vec{h}\|} = 0.$$

Now: call $\vec{h} = (u,v)$:

$$f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0}) \cdot \vec{h} = f(u,v) - 0 - (0,0)\cdot(u,v) = f(u,v).$$

We have therefore to prove that

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\|(u,v)\|} = 0.$$

This is a limit in $\mathbb{R}^2$ that we will compute by using the methods of previous chapter. Notice that

$$\frac{f(u,v)}{\|(u,v)\|} = \frac{\frac{u^2v^2}{u^2+v^2}}{\sqrt{u^2+v^2}} = \frac{u^2v^2}{(u^2+v^2)^{3/2}} \overset{u=\rho\cos\theta,\ v=\rho\sin\theta}{=} \frac{\rho^4(\cos\theta)^2(\sin\theta)^2}{\rho^3} = \rho(\cos\theta)^2(\sin\theta)^2,$$

hence

$$\left|\frac{f(u,v)}{\|(u,v)\|}\right| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0.$$

This finishes the exercise and prove that f is differentiable in 0_2 and $f'(0_2) = (0,0)$. $\square$

A useful differentiability test is the following:

Proposition 2.2.4

Let $\vec{F} = (f_1,\dots,f_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, D open. If

$$\partial_j f_i \in \mathscr{C}(D),\ \forall i,j, \implies \exists f \text{ is differentiable at any } \vec{x} \in D$$

A function $\vec{F}$ fulfilling this hypothesis is called a $\mathscr{C}^1(D)$ function.

EXAMPLE 2.2.5. *Discuss differentiability for the function of Example 2.2.3.*

SOL. — At $(x, y) \neq \vec{0}$ we may say that

$$\partial_x f(x, y) = \partial_x \frac{x^2 y^2}{x^2 + y^2} = y^2 \frac{2x(x^2 + y^2) - x^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2} \in \mathscr{C}(\mathbb{R}^2 \backslash \{\vec{0}\}),$$

and, similarly, $\partial_y f(x, y) \in \mathscr{C}(\mathbb{R}^2 \backslash \{\vec{0}\})$. Thus $f \in \mathscr{C}^1(\mathbb{R}^2 \backslash \{\vec{0}\})$. At $\vec{0}$, we computed above $\partial_x f(0, 0) = 0$. Thus, to apply previous proposition we need to check if $\partial_x f(x, y)$ is continuous at $\vec{0}$, that is, if

$$0 = \partial_x f(0, 0) = \lim_{(x,y) \to \vec{0}} \partial_x f(x, y) = \lim_{(x,y) \to \vec{0}} \frac{2xy^4}{(x^2 + y^2)^2}.$$

Introducing polar coordinates,

$$\left| \frac{2xy^4}{(x^2 + y^2)^2} \right| = \left| \frac{2\rho^5 \cos\theta (\sin\theta)^4}{\rho^4} \right| = 2\rho |\cos\theta| (\sin\theta)^4 \leqslant 4\rho \longrightarrow 0, \text{ when } \rho \longrightarrow 0.$$

Thus $\partial_x f(x, y)$ is continuous at $\vec{0}$. Similarly $\partial_y f(x, y)$ is continuous at $\vec{0}$. In conclusion: $f \in \mathscr{C}^1(\mathbb{R}^2)$, hence it is differentiable on $\mathbb{R}^2$ and $f'(x, y) = \nabla f(x, y)$ at every $(x, y) \in \mathbb{R}^2$. □

Differentiability is stronger than directional differentiability. This follows as by product of the following

Proposition 2.2.6

If $\vec{F}$ is differentiable at $\vec{x}$, then it is therein continuous.

PROOF. By $\vec{F}(\vec{y}) = \vec{F}(\vec{x}) + \vec{F}'(\vec{x})(\vec{y} - \vec{x}) + o(\vec{y} - \vec{x}) \longrightarrow \vec{F}(\vec{x})$, when $\vec{y} \longrightarrow \vec{x}$. □

Rules of calculus of differentials basically follows the same rules of those of ordinary calculus. For instance

$$(\vec{F} + \vec{G})'(\vec{x}) = \vec{F}'(\vec{x}) + \vec{G}'(\vec{x}).$$

provided $\vec{F}$ and $\vec{G}$ are differentiable at $\vec{x}$. Similarly it holds the important

Proposition 2.2.7: chain rule

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{G} : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$, be such that

i) $\exists \vec{F}'(\vec{x})$;

ii) $\exists \vec{G}'(\vec{F}(\vec{x}))$.

Then

$$\exists\ (\vec{G} \circ \vec{F})'(\vec{x}) = \vec{G}'(\vec{F}(\vec{x}))\vec{F}'(\vec{x}). \tag{2.2.4}$$

A special case of (2.2.4) is the following: suppose that we want to compute

$$\frac{d}{dt} f(\vec{\gamma}(t)), \text{ where } \vec{\gamma} : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d, \quad f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}.$$

For example, in physics f represents the physical energy of the system, $\vec{\gamma}$ a trajectory of motion. Then $f(\vec{g}(t))$ represents the variation of energy along the trajectory of motion. Calculating its ordinary derivative, we aim to measure the *rate of variation* of this quantity. Assuming that all the required hypotheses are fulfilled, we have the following.

$$\frac{d}{dt}f(\vec{\gamma}(t)) = f'(\vec{\gamma}(t))\vec{\gamma}'(t) = [\partial_1 f(\vec{\gamma}(t)) \ \ldots \ \partial_d f(\vec{\gamma}(t))] \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_d'(t) \end{bmatrix} = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t). \tag{2.2.5}$$

This quantity is also called **total derivative of f along $\vec{\gamma}$**.

2.3. Extrema

Roughly speaking, at the Min/Max points, the derivative vanishes. Under suitable specifications, this is a true fact known as *Fermat's Theorem*. Specifications concern the nature of the Min/Max point: they can be just *local* and must be in the interior of the domain. Let us first introduce the concept of *local Min/Max point*:

Definition 2.3.1

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We say that a point $\vec{x}_0 \in D$ is a **local minimum point for f** if

$$\exists B(\vec{x}_0, r] \ : \ f(\vec{x}_0) \leqslant f(\vec{x}), \ \forall \vec{x} \in B(\vec{x}_0, r] \cap D.$$

Local maximum point for f is defined similarly.

The most important fact of this Section is the

Theorem 2.3.2: Fermat

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\vec{x}_0 \in \mathrm{Int}(D)$ be a local min/max. If f is differentiable at $\vec{x}_0$ then $\nabla f(\vec{x}_0) = \vec{0}$.

Proof. We consider the case of $\vec{x}_0$ a local minimum, the case of a local maximum being similar. Thus, we assume that

$$\exists \ B(\vec{x}_0, r], \ : \ f(\vec{x}) \leqslant f(\vec{x}_0), \ \forall \vec{x} \in B(\vec{x}_0, r] \cap D.$$

Since $\vec{x}_0 \in \mathrm{Int}(D)$, we may assume directly that $B(\vec{x}_0, r] \subset D$. Now, consider the section of f on a straight line passing by $\vec{x}_0$. Fixed a direction $\vec{v}$, this is described by

$$\vec{\gamma}(t) = \vec{x}_0 + t\vec{v}.$$

Notice that

$$\vec{\gamma}(t) \in B(\vec{x}_0, r], \iff r \geqslant \|(\vec{x}_0 + t\vec{v} - \vec{x}_0\| = \|t\vec{v}\| = |t|\|\vec{v}\|, \iff |t| \leqslant \frac{r}{\|\vec{v}\|}.$$

Thus, if $g(t) := f(\vec{\gamma}(t))$, we have

$$g(t) \geqslant g(0) = f(\vec{x}_0), \ \forall |t| \leqslant \frac{r}{\|\vec{v}\|}.$$

In particular, g ha a minimum at $t = 0$. According to the real variable Fermat's Theorem, $g'(0) = 0$. But, recalling of total derivative formula (2.2.5),

$$g'(t) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t),$$

by which

$$0 = \nabla f(\vec{x}_0) \cdot \vec{v}.$$

This relation holds whatever is $\vec{v} \in \mathbb{R}^d$. Choosing $\vec{v} = \nabla f(\vec{x}_0)$ we read

$$0 = \nabla f(\vec{x}_0) \cdot \nabla f(\vec{x}_0) = \|\nabla f(\vec{x}_0)\|^2, \implies \nabla f(\vec{x}_0) = \vec{0}. \quad \square$$

Fermat's theorem is a fundamental tool for the search of min/max points. However, exactly as in the case of one real variable functions, some important remarks have to be done to have clear some features of the theorem. The first one is the following: $\nabla f(\vec{x}_0) = \vec{0}$ **does not mean necessarily that $\vec{x}_0$ is a min/max point**.

Example 2.3.3. *Let $f(x, y) = x^2 - y^2$ on $D = \mathbb{R}^2$. Determine points where $\nabla f(x, y) = \vec{0}$ and say if they are min/max points.*

Sol. — Clearly, $\nabla f(x, y) = (2x, -2y)$, therefore $\nabla f(0, 0) = (0, 0)$. However $(0, 0)$ is not a minimum nor a maximum. Indeed, notice that

$$f(x, 0) = x^2 \geqslant 0 = f(0, 0), \ \forall x \in \mathbb{R}.$$

Thus $(0, 0)$ is a minimum for the $x-$section. However,

$$f(0, y) = -y^2 \leqslant 0 = f(0, 0), \ \forall y \in \mathbb{R},$$

thus $(0, 0)$ is a max for the $y-$section. By this, $(0, 0)$ is neither a local min nor local max for f. Indeed, whatever is $B(\vec{0}, r]$, there are points $(x, 0), (0, y) \in B(\vec{0}, r]$, then

$$f(0, y) = -y^2 < 0 = f(0, 0) < x^2 = f(x, 0). \quad \square$$

For this reason we introduce the

Definition 2.3.4

A point $\vec{x}_0$ such that $\nabla f(\vec{x}_0) = \vec{0}$ is called **stationary point** for f.

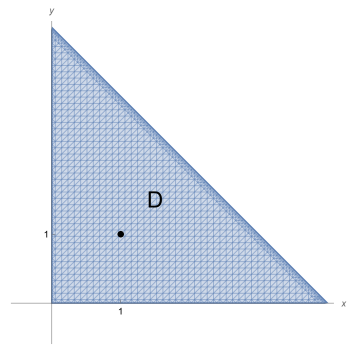
Condition $\nabla f(\vec{x}_0) = \vec{0}$ is the same of condition $f'(x_0) = 0$ for functions of one real variable. As in this case, $\nabla f(\vec{x}_0) = \vec{0}$ does not imply that $\vec{x}_0$ is a minimum or a maximum. For example, if

$$f(x, y) = x^2 - y^2,$$

easily $\nabla f(0, 0) = (0, 0)$ but $(0, 0)$ is neither a local min (because $f(0, y) = -y^2 \leqslant f(0, 0)$) nor a local max (because $f(x, 0) = x^2 \geqslant f(0, 0)$). Furthermore, $f(\vec{x}_0) = \vec{0}$ might give an information about possible min/max points only for $\vec{x}_0 \in \mathrm{Int}(D)$. We illustrate these issues with the following example.

Example 2.3.5. *Determine min/max for $f(x, y) = x + y - xy$ on $D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leqslant y \leqslant 3 - x\}$.*

Sol. — Notice first that $f \in \mathscr{C}(D)$ and D is manifestly closed (defined through a large inequality involving a continuous function) and bounded (trivial), thus compact. According to Weierstrass' theorem, min/max for f on D exist.



To determine them we may proceed as follows. Let $(x, y) \in D$ be a min/max point for f. Then

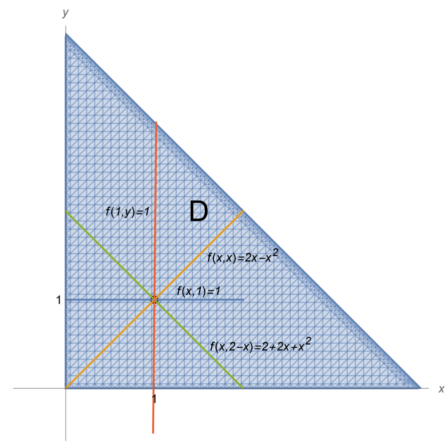
- if $(x, y) \in \mathrm{Int}(D) = \{0 < y < 4 - x\}$, according to Fermat's theorem, $\nabla f(x, y) = \vec{0}$. Since $\nabla f(x, y) = (1 - y, 1 - x)$, we have
$$\nabla f(x, y) = (0, 0), \iff \begin{cases} 1 - y = 0, \\ \\ 1 - x = 0, \end{cases} \iff (x, y) = (1, 1).$$
Since $(1, 1) \in D$, this point is a possible candidate to be min/max point for f.
- if $(x, y) \notin \mathrm{Int}(D)$, that is $(x, y) \in \partial D$, we cannot say $\nabla f(x, y) = \vec{0}$. However, it is easy to solve the puzzle. Notice first that
$$\partial D = \{(x, 0) : 0 \leqslant x \leqslant 4\} \cup \{(0, y) : 0 \leqslant y \leqslant 4\} \cup \{(x, 4 - x) : 0 \leqslant x \leqslant 4\} =: \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$
On Γ_1: $f(x, 0) = x$ with $x \in [0, 4]$, thus the min point of f on Γ_1 is achieved at $x = 0$ (point $(0, 0)$) while max point of f on Γ_1 is achieved at $x = 4$ (point $(4, 0)$). These points are, for now, min/max for f just on Γ_1.
  On Γ_2 we have a similar discussion because $f(0, y) = y$ with $y \in [0, 4]$. Thus $(0, 0)$ is a min for f on Γ_2 while $(0, 4)$ is a max for f on Γ_2.
  On Γ_3, $f(x, 4 - x) = x + (4 - x) - x(4 - x) = 4 - 4x + x^2 = (x - 2)^2$, and this quantity is minimum for $x = 2$ (thus f has a min on Γ_3 at point $(2, 2)$), maximum at $x = 0, 4$ (thus f has a maximum on Γ_3 at points $(4, 0)$ and $(0, 4)$).

We have now all the ingredients to draw the conclusion:

- max points: candidates to be max points for f on D are $(1, 1)$ (stationary point in $\mathrm{Int}(D)$), $(4, 0)$, $(0, 4)$. Since
$$f(1, 1) = 1, \quad f(4, 0) = f(0, 4) = 4,$$
we conclude that $(4, 0)$ and $(0, 4)$ are max points for f on D.
- min points: candidates to be min points for f on D are $(1, 1)$, $(0, 0)$ and $(2, 2)$. Since
$$f(1, 1) = 1, \; f(0, 0) = 0, \; f(2, 2) = 0,$$
we conclude that $(0, 0)$ and $(2, 2)$ are minimum points for f on D.

Someone may ask: what is, then, point $(1, 1)$ for f? Let us give a look at few sections of f passing through point $(1, 1)$. We will consider the following ones:
$$y = 1, \quad x = 1, \quad y = x, \quad y = 2 - x.$$

We have $f(x,1) = x + 1 - x \cdot 1 = 1$, that is f is constant, and similarly $f(1,y) \equiv 1$. On $y = x$, we have $f(x,x) = 2x - x^2 =: g(x)$. Since $g'(x) = 2 - 2x \geqslant 0$ iff $x \leqslant 1$, we see that $x = 1$ is a max point for g, thus $(1,1)$ is max for f along curve $y = x$. This might lead to think that $(1,1)$ is perhaps a (local) maximum. However, when we consider $y = 2 - x$, we see that $f(x, 2-x) = x+(2-x)-x(2-x) = 2-2x+x^2 = 1+(x-1)^2$ from which we clearly see that $x = 1$ is a minimum for this quantity. That is, $(1,1)$ is a min point for f along $y = 2 - x$. Since we have two sections along which the same point $(1,1)$ is in one case a minimum, in the other a maximum for f, we conclude that $(1,1)$ is neither a local minimum nor a local maximum for f. □



Previous example suggests a definition:

Definition 2.3.6

Let $\vec{x}_0$ be a stationary point for f. If there are two different curves $\vec{\gamma}_1, \vec{\gamma}_2 : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d$ passing through point $\vec{x}_0$ (that is $\vec{\gamma}_1(t_0) = \vec{\gamma}_2(t_0) = \vec{x}_0$ for some $t_0 \in I$) such that $\vec{x}_0$ is max point for f along $\vec{\gamma}_1$ and min point for f along $\vec{\gamma}_2$, that is

$$f(\vec{\gamma}_1(t)) < f(\vec{x}_0) < f(\vec{\gamma}_2(t)),\ \forall t \in I \backslash \{t_0\},$$

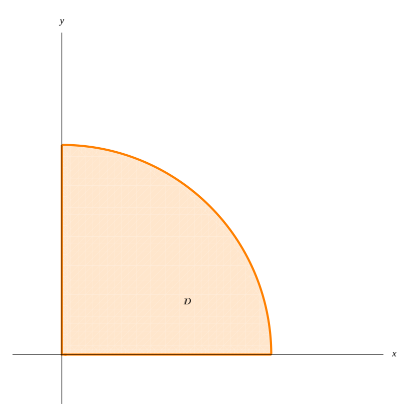
then $\vec{x}_0$ is called **saddle point** for f.

Example 2.3.7. *Let*

$$f(x,y) = xye^{x^2+y^2},\quad (x,y) \in D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant 1\}.$$

Draw D. What can you say about D: is open? closed? compact? connected? Show that f admits global extrema on D and find these points. Finally, determine $f(D)$.

Sol. — Being D defined by large inequalities it is closed. It is not open because the points on the boundary belongs to D. Moreover, D is a subset of unit disk, therefore is bounded, hence is compact. A picture of D is easy (see figure). Let us now discuss min/max of f.
Existence. Since f is continuous and D compact, Weierstrass's thm says that f has global min and max.
Determination. Let $(x,y) \in D$ be a min/max point (whose existence has been ensured above). We have the alternative either $(x,y) \in \mathrm{Int}(D)$ (hence, according to Fermat's theorem, $\nabla f(x,y) = \vec{0}$) or $(x,y) \in \partial D$ (and, in this case, $\nabla f(x,y)$ is not necessarily null).



Now,

$$\nabla f(x,y) = \left(e^{x^2+y^2}(y+2x^2y), e^{x^2+y^2}(x+2y^2x)\right) = 0, \iff \begin{cases} y(1+2x^2) = 0, \\ \\ x(1+2y^2) = 0, \end{cases} \iff x = y = 0.$$

The unique stationary point for f is $(0,0)$ which is not, however, in the $\mathrm{Int}(D)$. This means that *there are not stationary points for f in* $\mathrm{Int}(D)$. In particular, min/max points are certainly on ∂D.

So consider $(x,y) \in \partial D$. We have

$$\partial D = \{(x,0) \ : \ 0 \leqslant x \leqslant 1\} \cup \{(0,y) \ : \ 0 \leqslant y \leqslant 1\} \cup \left\{(x,y) \ : \ x^2+y^2 = 1, \ x \geqslant 0, \ y \geqslant 0\right\} =: \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$

On Γ_1 we have $f(x,0) = 0$, so f is constant; on Γ_2 we have the same $f(0,y) = 0$. On Γ_3, noticed that we may also write

$$\Gamma_3 = \left\{(\cos\theta, \sin\theta) \ : \ \theta \in \left[0, \frac{\pi}{2}\right]\right\},$$

we have

$$f(\cos\theta, \sin\theta) = (\cos\theta)(\sin\theta)e^1 = \frac{e}{2}\sin(2\theta).$$

Clearly, this quantity is maximum when $2\theta = \frac{\pi}{2}$, that is $\theta = \frac{\pi}{4}$, that is at point $\frac{1}{\sqrt{2}}(1,1)$, minimum when $\theta = 0, \frac{\pi}{2}$, that is at points $(1,0)$ and $(0,1)$. In conclusion, candidates min points are points $(x,0)$ $(x \in [0,1])$, $(0,y)$ $(y \in [0,1])$ where f is constantly equal to 0, hence all these points are minimum points. Candidates max points are, again, points $(x,0)$, $(0,y)$ (for $x, y \in [0,1]$) and $\frac{1}{\sqrt{2}}(1,1)$, where f takes value $\frac{e}{2}$. We conclude that there is a unique max point and it is $\frac{1}{\sqrt{2}}(1,1)$.

Finally: because D is clearly connected, according to the intermediate values theorem $f(D)$ is an interval, and precisely $f(D) = [0, \frac{e}{2}]$. □

So far we have seen examples of optimization problems on compact domains. Let us see an example of optimization problem on a non compact domain.

Example 2.3.8. *Determine min/max (if any) of $f(x,y,z) := x^2+y^2+z^2-xy+z$ on $D := \mathbb{R}^3$. What about $f(D)$?*

Sol. — Here $D = \mathbb{R}^3$ is clearly closed and unbounded, therefore ordinary Weierstrass' theorem does not apply. Clearly $f \in \mathscr{C}(\mathbb{R}^3)$. Let us check if the limit at ∞_3 exists. Notice that $f(x,0,0) = x^2 \longrightarrow +\infty$ if $\|(x,0,0)\| \longrightarrow +\infty$, thus if $\lim_{(x,y,z)\to\infty_3} f(x,y,z)$ exists it must be $= +\infty$. To prove this, let us pass to spherical coordinates,

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi. \end{cases}$$

We have

$$f(x,y,z) = \rho^2 - \rho^2\cos\theta\sin\theta(\sin\varphi)^2 + \rho\cos\varphi = \rho^2\left(1 - \frac{1}{2}\sin(2\theta)(\sin\varphi)^2\right) + \rho\cos\varphi.$$

Therefore,

$$f(x,y,z) \geqslant \rho^2\left(1-\frac{1}{2}\right) - \rho = \frac{\rho^2}{2} - \rho \longrightarrow +\infty, \text{ when } \rho \longrightarrow +\infty.$$

This proves that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. According to Proposition 1.5, f has global minimum on $\mathbb{R}^3$ (but, of course, there is no global maximum being f upper unbounded). This proves existence, let us now pass to the determination.

Let $(x, y, z) \in \mathbb{R}^3$ be a min point for f. Since clearly $(x, y, z) \in \mathrm{Int}(\mathbb{R}^3) = \mathbb{R}^3$, according to Fermat's theorem $\nabla f(x, y, z) = \vec{0}$. We have

$$\nabla f(x,y,z) = (2x-y, 2y-x, 2z-1) = \vec{0}, \iff \begin{cases} 2x-y=0, \\ 2y-x=0, \\ 2z-1=0, \end{cases} \iff \begin{cases} x=0, \\ y=0, \\ z=\frac{1}{2}. \end{cases}$$

Thus, the unique possible minimum is $(0, 0, \frac{1}{2})$ and since minimum exists (previous part), this implies that $(0, 0, \frac{1}{2})$ is the minimum for f on $\mathbb{R}^3$.

Finally, since $D = \mathbb{R}^3$ is clearly connected, $f(D)$ is an interval and this is $[f(0, 0, \frac{1}{2}), +\infty[= \left[\frac{1}{2}, +\infty\right[$. □

2.4. Constrained Optimization

Many applied problems can be formalized as the maximization/minimization of a certain quantity (function) f of several variables over certain *constraints* on the variables. For instance, consider the problem of searching for the parallelepiped with maximum volume among those with fixed surface S. Formalizing this we have

$$\max_{x,y,z>0\,:\,2(xy+yz+xz)=S} xyz.$$

A general form for this problem is

$$\min/\max_{\mathscr{M}} f(x_1, \dots, x_d), \quad \text{on } \mathscr{M} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \;:\; g_1(x_1, \dots, x_d) = 0, \dots, g_m(x_1, \dots, x_d) = 0\}.$$

The method developed in the previous section does not work in this context. Indeed, the optimization domain $\mathscr{M}$ has no interior points in general. Thus, at min/max points (if any), ∇f is not necessarily $= \vec{0}$.

To understand the strategy, let us consider a "downsized" version of the problem posed above, that is the problem of searching for the rectangle with maximum area with fixed perimeter P. Formally, we want to determine

$$\max_{(x,y)\in]0,+\infty[\,:\,2(x+y)=P} xy.$$

Here, the optimization domain is $\{y = \frac{P}{2} - x\}$, that is a straight line, clearly a set with no interior points in the plane $\mathbb{R}^2$. In this simple case, we can easily solve the problem. Since $y = \frac{P}{2} - x$, we have that

$$xy = x\left(\frac{P}{2} - x\right).$$

Thus, maximizing xy is the same of maximizing $x\left(\frac{P}{2} - x\right)$ where now x is "unconstrained" (the unique condition is $x \in]0, +\infty[$). In other words

$$\max_{(x,y)\in]0,+\infty[\,:\,2(x+y)=P} xy = \max_{x\in]0,\frac{P}{2}[} x\left(\frac{P}{2} - x\right),$$

which is an elementary problem. This example suggests a general idea. Imagine we have to solve

$$\min/\max_{g(x,y)=0} f(x, y).$$

Suppose that we can use the equation $g(x, y) = 0$ to "extract" $y = \phi(x)$ or, which is the same, $x = \psi(y)$. Then,

$$\min/\max_{g(x,y)=0} f(x,y) = \min/\max_{x} f(x, \phi(x)) = \min/\max_{y} f(\psi(y), y).$$

The problem is that, in general, we might not be able to explicit y or x from an equation $g(x, y) = 0$. The following informal argument shows that this is not necessarily needed. Indeed, at min/max points for $f(x, \phi(x))$ we have

$$\frac{d}{dx} f(x, \phi(x)) = 0.$$

By the chain rule, this means

$$0 = \partial_x f(x, \phi(x)) + \partial_y f(x, \phi(x))\phi'(x).$$

Now, since $y = \phi(x)$ verifies $g(x, y) = 0$, then

$$g(x, \phi(x)) \equiv 0, \implies 0 = \frac{d}{dx} g(x, \phi(x)) = \partial_x g(x, \phi(x)) + \partial_y g(x, \phi(x))\phi'(x),$$

from which, *provided* $\partial_y g(x, y) \neq 0$,

$$\phi'(x) = -\frac{\partial_x g(x, \phi(x))}{\partial_y g(x, \phi(x))}.$$

Thus

$$0 = \partial_x f(x, \phi(x)) + \partial_y f(x, \phi(x)) \left(-\frac{\partial_x g(x, \phi(x))}{\partial_y g(x, \phi(x))} \right),$$

that is, rearranging terms,

$$(\partial_x f, \partial_y f) \cdot (\partial_y g, -\partial_x g) = 0.$$

This means

$$\nabla f \perp (\partial_y g, -\partial_x g), \iff \nabla f \parallel (\partial_x g, \partial_y g) = \nabla g.$$

The same conclusion can be obtained if from $g(x, y) = 0$ we can explicit $x = \psi(y)$, provided $\partial_x g \neq 0$. Making a formal proof we obtain the following:

Theorem 2.4.1: Lagrange multilplier

Let $g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that $\nabla g \neq \vec{0}$ on $\mathscr{M} := \{g = 0\}$. Then, if (x, y) is a min/max point for f on $\mathscr{M}$, necessarily

$$\exists \lambda \in \mathbb{R} \ : \ \nabla f(x, y) = \lambda \nabla g(x, y).$$

Example 2.4.2. *Determine*

$$\min/\max_{x^2+y^2=1} (x + y).$$

Sol. — Let $f(x, y) = x + y$ and $\mathscr{M} = \{x^2 + y^2 = 1\} = \{x^2 + y^2 - 1 = 0\} =: \{g = 0\}$. Clearly, $\mathscr{M}$ is closed and bounded, hence compact, $f \in \mathscr{C}$, min and max of f on $\mathscr{M}$ are ensured by Weierstrass' theorem.

To determine these points, we apply Lagrange multiplier's theorem. We first check that $\nabla g \neq \vec{0}$ on $\mathscr{M}$. Indeed,

$$\nabla g = (2x, 2y) = \vec{0}, \iff (x, y) = (0, 0) \notin \mathscr{M},$$

thus $\nabla g \neq \vec{0}$ on $\mathscr{M}$. Now, if (x, y) is a min/max point, we have

$$\nabla f(x, y) = \lambda \nabla g(x, y).$$

Since $\nabla f = (1, 1)$, this means

$$(1, 1) = \lambda(2x, 2y), \iff \begin{cases} 1 = 2\lambda x, \\ 1 = 2\lambda y, \end{cases} \iff (x, y) = \left(\frac{1}{2\lambda}, \frac{1}{2\lambda} \right).$$

Now, this point must belong to $\mathscr{M}$, that is

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1, \iff 2\lambda^2 = 1, \iff \lambda = \pm\frac{1}{\sqrt{2}},$$

that leads points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. These are candidates to be min/max points. It is now sufficient to notice that

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}, \quad f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2},$$

from which we see that the first point is a max point, the latter is a min point. □

We give a specific name to the technical requirement on g:

Definition 2.4.3

We say that $g = g(x, y)$ is a **submersion** on $S \subset \mathbb{R}^2$ if $\nabla g(x, y) \neq \vec{0}$ for every $(x, y) \in S$.

Lagrange multiplier λ is unnecessary to determine points where $\nabla f = \lambda \nabla g$. Indeed:

$$\nabla f(x, y) = \lambda \nabla g(x, y), \iff \operatorname{rank}\begin{bmatrix} \nabla f(x, y) \\ \\ \nabla g(x, y) \end{bmatrix} = 1, \iff \det\begin{bmatrix} \nabla f(x, y) \\ \\ \nabla g(x, y) \end{bmatrix} = 0.$$

Example 2.4.4. *Find points of the ellipse $x^2 + 2y^2 - xy = 9$ at min/max distance to the origin.*

Sol. — We have to minimize/maximize the distance to the origin, that is the function

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Because of the properties of the root, to minimize this function or just $x^2 + y^2$ is the same (it produces the same points but of course not the same values!) being $\sqrt{x^2 + y^2}$ min/max iff $x^2 + y^2$ it is, we replace the previous f with

$$f(x, y) = x^2 + y^2,$$

which is easier to be managed. Let also $g(x, y) := x^2 + 2y^2 - xy - 9$ in such a way we have to maximize/minimize f on $\{g = 0\}$.

Existence. Clearly $f \in \mathscr{C}(\mathbb{R}^2)$. The optimization domain is $\{g = 0\}$ which is clearly closed being defined by en equality involving a continuous function $g \in \mathscr{C}$. Let us check that $\{g = 0\}$ is also bounded. Recalling that

$$xy \leqslant \frac{1}{2}(x^2 + y^2),$$

then, for $(x, y) \in \{g = 0\}$, we have

$$x^2 + 2y^2 = 9 + xy \leqslant 9 + \frac{1}{2}(x^2 + y^2), \implies \frac{1}{2}x^2 + \frac{3}{2}y^2 \leqslant 9, \implies \frac{1}{2}x^2, \frac{3}{2}y^2 \leqslant 9,$$

by which $x^2 \leqslant 18$ (hence $|x| \leq \sqrt{18}$) and $y^2 \leqslant 6$ (that is $|y| \leqslant \sqrt{6}$). In any case both x, y are bounded hence $\{g = 0\}$ is bounded. According to Weierstrass' theorem, f admits both min/max on $\{g = 0\}$.

Determination. By the previous argument we know that min/max points for f exist. Let's see if we can apply Lagrange's multipliers theorem. We need to check if g is a submersion on $\{g = 0\}$. To this aim let's see where g is not a submersion. This happens iff

$$\nabla g = 0, \iff (2x-y, 4y-x) = (0,0), \iff \begin{cases} 2x - y = 0, \\ \\ 4y - x = 0, \end{cases} \iff x = y = 0.$$

Therefore g is not a submersion at point $(0,0)$ and since $g(0,0) \neq 0$ we conclude that $(0,0) \notin \{g = 0\}$. Hence g is a submersion on $\{g = 0\}$.

According to Lagrange's theorem, at a min/max point we must have $\nabla f = \lambda \nabla g$ or, as noticed above,

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 2x & 2y \\ 2x - y & 4y - x \end{bmatrix} = 2x(4y-x) - 2y(2x-y) = 2(y^2 - x^2) + 4xy.$$

This can be rewritten as

$$(x+y)^2 - 2x^2 = 0, \iff (x+y)^2 = 2x^2, \iff x + y = \pm\sqrt{2}x, \iff y = (\pm\sqrt{2} - 1)x.$$

Therefore we have points $(x, (\pm\sqrt{2}-1)x)$. Of course we have to look at those of them that belongs to $\mathscr{M}$:

$$(x, (\sqrt{2}-1)x) \in \mathscr{M}, \iff x^2 + 2(\sqrt{2}-1)^2 x^2 - (\sqrt{2}-1)x^2 = 9, \iff (8 - 5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8-5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right)$ (same sign, 2 points). Similarly

$$(x, (-\sqrt{2}-1)x) \in \mathscr{M}, \iff x^2 + 2(-\sqrt{2}-1)^2 x^2 - (-\sqrt{2}-1)x^2 = 9, \iff (8+5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8+5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right)$ (same sign, two points). Now, being

$$f\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right) = \frac{36 - 18\sqrt{2}}{8 - 5\sqrt{2}} > f\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right) = \frac{36 + 18\sqrt{2}}{8 + 5\sqrt{2}}$$

we have that the first points are max for f, the latter are min. □

Lagrange multiplier's theorem extends to the more general case of problem

$$\underset{g(x_1,\ldots,x_d)=0}{f} (x_1, \ldots, x_d).$$

Theorem 2.4.5: Lagrange multilplier

Let $g : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be a submersion on $\mathscr{M} := \{g = 0\}$ (that is, $\nabla g \neq \vec{0}$ on $\mathscr{M}$). Then, if $\vec{x}$ is a min/max point for f on $\mathscr{M}$, necessarily

$$\exists \lambda \in \mathbb{R} \ : \ \nabla f(\vec{x}) = \lambda \nabla g(\vec{x}).$$

Notice that, in this case

$$\nabla f = \lambda \nabla g, \iff \operatorname{rank} \begin{bmatrix} \nabla f \\ \\ \nabla g \end{bmatrix} = 1.$$

This matrix is now a $2 \times d$ matrix. Its rank is 1 iff *all* 2×2 *sub-determinants are* $= 0$.

EXAMPLE 2.4.6. *Solve the isoperimetric problem*

$$\max_{(x,y,z)\in\mathbb{R}^3:\ x,y,z\geqslant 0,\ 2(xy+yz+xz)=S} xyz.$$

SOL. — Let $f(x,y,z) := xyz$, clearly $f \in \mathscr{C}$. Let $\mathscr{M} := (x,y,z) \in \mathbb{R}^3 :\ x,y,z \geqslant 0,\ 2(xy+yz+xz) = S$, thus $\mathscr{M}$ is closed. Unfortunately, $\mathscr{M}$ is unbounded (for example, $(0,y,z) \in \mathscr{M}$ iff $yz = \frac{A}{2}$, which is an hyperbola. However, certainly the maximum of xyz cannot be attained if one of the three coordinates is big. Indeed, if $x = K$, then being $xy, xz \leqslant \frac{S}{2}$ we would have $y,z \leqslant \frac{S}{2K}$, thus $xyz \leqslant K\frac{S^2}{4K^2} = \frac{S^2}{4K}$. Since the greater is K, the lower is this value, we conclude that the maximum of xyz is attained for $0 \leqslant x,y,z \leqslant K$ with K big enough. Thus, search of max may be restricted to a compact domain, wherein it is ensured by Weierstrass' theorem.

To determine this maximum, we apply the Lagrange multiplier's theorem. First, let us check that $g = 2(xy+yz+xz) - A$ is a submersion. We have

$$\nabla g = (2(y+z), 2(x+z), 2(x+y)) = \vec{0}, \iff \begin{cases} y+z=0, \\ x+z=0, \\ x+y=0, \end{cases} \iff (x,y,z) = (0,0,0) \notin \mathscr{M}.$$

Thus $\nabla g \neq \vec{0}$ on $\mathscr{M}$.

Let now (x,y,z) be a max point for f. According to Lagrange's theorem,

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z), \iff \operatorname{rank}\begin{bmatrix} yz & xz & xy \\ 2(y+z) & 2(x+z) & 2(x+y) \end{bmatrix} = 1.$$

This happens iff all the 2×2 sub-determinants of this last matrix vanish, that is

$$\begin{cases} yz(x+z) - xz(y+z) = 0, \\ yz(x+y) - xy(y+z) = 0, \\ xz(x+y) - xy(x+z) = 0, \end{cases} \iff \begin{cases} z^2(y-x) = 0, \\ y^2(z-x) = 0, \\ x^2(z-y) = 0. \end{cases}$$

The first poses either $z = 0$ or $y = x$. In the first case, the remaining equations reduces to $xy = 0$, that is either $x = 0$ or $y = 0$. This means points $(0,y,0)$ and $(x,0,0)$, none of which can be maximum point since in both cases volume=0. In the second case, $y = x$, second and third equations reduce to $x^2(z-x) = 0$, that is either $x = 0$ (then solutions $(0,0,z)$, none of which is maximum) or $z = x$. Thus the unique possible candidates are (x,x,x). Imposing the perimeter condition, $3x^2 = \frac{S}{2}$, $x = \sqrt{\frac{S}{6}}$, and we conclude that *the volume is maximum when the parallelepiped is a cube of side* $x = \sqrt{\frac{S}{6}}$. □

EXAMPLE 2.4.7. *A segment of length L is divided into n parts $x_1,\dots,x_n$. Find the maximum of $x_1\cdots x_n$. Deduce by this the classical inequality*

$$\sqrt[n]{x_1\cdots x_n} \leqslant \frac{x_1+\cdots+x_n}{n}, \quad \forall x_1,\dots,x_n \geqslant 0.$$

SOL. — We have to find

$$\max_{x_1+\cdots+x_n=L,\ x_1,\dots,x_n>0} x_1\cdots x_n.$$

Let $\mathscr{M} := \{x_1+\cdots+x_n = L \ :\ x_1,\dots,x_n > 0\} = \{g = 0\}$, where $g = x_1+\cdots+x_n - L$ defined on $D =]0,+\infty[^n$. Clearly $g \in \mathscr{C}^1$ and since

$$\nabla g = (1,\dots,1) \neq 0,$$

g is a submersion on $\mathscr{M}$.

Let $f(x_1,\dots,x_n)=x_1\cdots x_n$. Clearly, $\mathscr{M}$ is closed. It is also bounded because, since $x_j\geqslant 0$, for all j, and $x_1+\dots+x_n=L$, we have $0\leqslant x_j\leqslant L$, $j=1,\dots,n$. Thus, f has min/max on $\mathscr{M}$. Stationary points of f on $\mathscr{M}$ must obey to

$$1=\operatorname{rank}\begin{bmatrix}\nabla f(x_1,\dots,x_n)\\ \\ \nabla g(x_1,\dots,x_n)\end{bmatrix}=\operatorname{rank}\begin{bmatrix}x_2\cdots x_n & x_1x_3\cdots x_n & \cdots & x_1\cdots x_{n-1}\\ \\ 1 & 1 & \cdots & 1\end{bmatrix}.$$

This is possible iff all the 2×2 sub determinants vanish. Choosing column i and j respectively we have

$$\det\begin{bmatrix}x_1\cdots x_{i-1}x_{i+1}\cdots x_n & x_1\cdots x_{j-1}x_{j+1}\cdots x_n\\ \\ 1 & 1\end{bmatrix}=x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j-x_i).$$

Therefore, $(x_1,\dots,x_n)\in\mathscr{M}$ is critic for f on $\mathscr{M}$ iff

$$x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j-x_i)=0,\ \forall i\neq j=1,\dots,n.$$

This produces points where a coordinate is null (hence $f=0$) and, if $x_j>0$ for any j, $x_i-x_j=0$ for all i,j, and this means that $(x_1,\dots,x_n)=(\alpha,\alpha,\dots,\alpha)$. Imposing that this belongs to $\mathscr{M}$ we find the point $(\frac{L}{n},\dots,\frac{L}{n})$ where $f>0$: therefore this is the maximum! The moral is

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1\cdots x_n=\left(\frac{L}{n}\right)^n.$$

In particular, recalling that $x_1+\ldots+x_n=L$, this can be rewritten as

$$x_1\cdots x_n\leqslant\left(\frac{x_1+\ldots+x_n}{n}\right)^n,\quad\Longleftrightarrow\quad\sqrt[n]{x_1\cdots x_n}\leqslant\frac{x_1+\ldots+x_n}{n},$$

that is just the classical inequality between arithmetic and geometric means. □

Example 2.4.8. *Among all the convex polygons inscribed into a circle, find those of maximum perimeter.*

Sol. — Let $r>0$ be the radius of the circle, $\theta_1,\dots,\theta_n$ the subsequent angles formed by the vertexes of the polygon. Then

$$\text{perimeter}\ =\ P(\theta_1,\dots,\theta_n)=\sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

Of course $0<\theta_j<2\pi$ and $\theta_1+\dots+\theta_n=2\pi$. Thus, we have to find

$$\max_{\theta_1+\ldots+\theta_n=2\pi,\ 0<\theta_j<2\pi,\ j=1,\ldots,n}\ \sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

Let

$$\mathscr{M}:=\{(\theta_1,\dots,\theta_n)\in]0,2\pi[^n:\ \theta_1+\dots+\theta_n=2\pi\}.$$

Arguing as in the previous example, we easily find stationary points of P on $\mathscr{M}$: they must fulfil

$$\operatorname{rank}\begin{bmatrix}r\cos\frac{\theta_1}{2} & \cdots & r\cos\frac{\theta_n}{2}\\ \\ 1 & \cdots & 1\end{bmatrix}=1,\ \Longleftrightarrow\ r\cos\frac{\theta_i}{2}=r\cos\frac{\theta_j}{2},\ \forall i,j,\ \Longleftrightarrow\ \theta_i=\theta_j,\ \forall i,j.$$

Therefore, the polygon with maximum perimeter has $\theta_1=\theta_2=\ldots=\theta_n=\frac{2\pi}{n}$, thus it is a regular polygon. □

2.5. Lagrange multipliers' theorem

Lagrange multiplier theorem extends to the case when there are more constraints, as for the problem

$$\min/\max_{\vec{x}\,:\,g_1(\vec{x})=0,\dots,g_m(\vec{x})=0} f(\vec{x}).$$

We introduce the

Definition 2.5.1

Let $\vec{G} := (g_1, \dots, g_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. We say that $\vec{G}$ is a **submersion** on S if

$$\nabla g_1(\vec{x}), \dots, \nabla g_m(\vec{x}) \text{ are linearly independent, } \forall \vec{x} \in S. \tag{2.5.1}$$

Remark 2.5.2. Since a vector is linearly independent iff it is different from zero, the previous definition encompasses the definition of submersion given in the previous section. □

Remark 2.5.3. Since $\nabla g_1(\vec{x}), \dots, \nabla g_m(\vec{x})$ are vectors of $\mathbb{R}^d$, they can be linearly independent only if $m \leqslant d$, otherwise this is impossible. □

Remark 2.5.4. A practical way to check condition (2.5.1) is the following:

$$\operatorname{rank}\begin{bmatrix} \nabla g_1(\vec{x}) \\ \nabla g_2(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = \operatorname{rank} \vec{G}'(\vec{x}) = m.$$

Since jacobian matrix $G'(\vec{x})$ is an $m \times d$ matrix (with $m \leqslant d$ as noticed in previous remark), rank $\vec{G}'(\vec{x}) = m$ iff *at least one $m \times m$ sub-determinant of $\vec{G}'(\vec{x})$ is not zero.* □

Theorem 2.5.5: Lagrange's multipliers theorem

Let $f = f(\vec{x}) \in \mathscr{C}^1$ and $\vec{G} = (g_1, \dots, g_m) \in \mathscr{C}^1$ be a submersion on $\mathscr{M} := \{g_1 = 0, \dots, g_m = 0\}$. Then, if $\vec{x} \in \mathscr{M}$ is a min/max point for f on $\mathscr{M}$,

$$\exists \lambda_1, \dots, \lambda_m \in \mathbb{R} \;:\; \nabla f(\vec{x}) = \lambda_1 \nabla g_1(\vec{x}) + \cdots + \lambda_d \nabla g_m(\vec{x}). \tag{2.5.2}$$

The points $\vec{x}$ where (2.5.2) holds are called **constrained stationary points**.

Condition (2.5.2) says that $\nabla f(\vec{x})$ is linearly dependent of $\nabla g_1(\vec{x}), \dots, \nabla g_m(\vec{x})$ and it involves "multipliers" $\lambda_1, \dots, \lambda_m$ which, however, are unnecessary. In fact, since $\vec{G} = (g_1, \dots, g_m)$ is a submersion in $\vec{x}$,

$$(2.5.2) \iff \operatorname{rank}\begin{bmatrix} \nabla f(\vec{x}) \\ \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = m.$$

Notice that the left matrix is an $(m+1)\times d$ matrix whose rank cannot be $m+1$. Therefore

$$(2.5.3)\qquad (2.5.2) \iff \text{all } (m+1)\times(m+1) \text{ subdeterminants of } \begin{bmatrix} \nabla f(\vec{x}) \\ \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} \text{ equals } 0.$$

EXAMPLE 2.5.6. *Let* $\mathscr{M} := \left\{(x,y,z)\in\mathbb{R}^3 \;:\; xy+z^2=1,\ x^2+y^2=1\right\}$. *i) Show that* $\mathscr{M}$ *is the non-empty zero set of a submersion in* $\mathscr{M}$. *ii) Say whether* $\mathscr{M}$ *is compact or less. iii) Find the points of* $\mathscr{M}$ *at the minimum / maximum distance from the origin.*

SOL. — i) Let us check that $\mathscr{M}\neq\emptyset$. Take a point of type (x,x,z). Imposing that it belongs to $\mathscr{M}$ we get $2x^2=1$, that is, $x=\pm\frac{1}{\sqrt{2}}$. By the first, then, $x^2+z^2=1$, that is, $z^2=1-x^2=1-\frac{1}{2}=\frac{1}{2}$, i.e. $z=\pm\frac{1}{\sqrt{2}}$. Therefore $(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})\in\mathscr{M}$ (all combinations of signs provided the sign of the first two coordinates is equal). This proves $\mathscr{M}\neq\emptyset$.

Define now $\vec{G}\equiv(g_1,g_2):=(xy+z^2-1, x^2+y^2-1)$. Clearly $\vec{G}\in\mathscr{C}^1$ and $\mathscr{M}=\{g_1=0,\ g_2=0\}$. We must discuss whether $\vec{G}$ is a submersion in $\mathscr{M}$. To this aim, let us find any point where $\vec{G}$ **is not** a submersion. This means

$$\operatorname{rank}\vec{G}'(x,y,z)<2, \iff \operatorname{rank}\begin{bmatrix} y & x & 2z \\ \\ 2x & 2y & 0\end{bmatrix}<2, \iff \begin{cases} 2(y^2-x^2)=0, \\ -4xz=0, \\ -4yz=0.\end{cases}$$

This produces the two cases

$$\begin{cases} x=0, \\ y^2=0, \iff y=0, \\ z\in\mathbb{R}\end{cases} \quad\text{or}\quad \begin{cases} z=0, \\ x^2-y^2=0, \iff y=x,\ \vee\ y=-x.\end{cases}$$

Therefore, $\vec{G}$ is not a submersion at points $(0,0,z)$, $z\in\mathbb{R}$ and $(x,x,0)$, $(x,-x,0)$, $x\in\mathbb{R}$. Clearly $(0,0,z)\notin\mathscr{M}$ for any $z\in\mathbb{R}$; moreover,

$$(x,x,0)\in\mathscr{M}, \iff \begin{cases} x^2=1, \\ 2x^2=1,\end{cases} \text{(impossible)},\quad (x,-x,0)\in\mathscr{M}, \iff \begin{cases} -x^2=1, \\ 2x^2=1,\end{cases} \text{(impossible)}.$$

We conclude that $\vec{G}$ is a submersion on $\mathscr{M}$.

ii) Since $\mathscr{M}=\{g_1=0, g_2=0\}$ and $g_1,g_2\in\mathscr{C}$, it follows that $\mathscr{M}$ is closed. It is also bounded because, by the second constraint, $x^2+y^2=1$ we deduce $|x|,|y|\leqslant 1$, and by the first

$$z^2=1-xy\leqslant 2, \implies |z|\leqslant\sqrt{2}.$$

iii) Distance from (x,y,z) to $(0,0,0)$ is $\sqrt{x^2+y^2+z^2}$. Because this quantity is min/max when $f(x,y,z)=x^2+y^2+z^2$ it is, we use this function to find min/max points. Since $\mathscr{M}$ is compact and f is continuous, Min/max exist by Weierstrass's theorem.

To determine these points, we may argue in the following way. Let $(x,y,z)\in\mathscr{M}$ be a min/max point for f. Since $\mathscr{M}=\{\vec{G}=\vec{0}\}$ and $\vec{G}$ is a submersion on $\mathscr{M}$, at (x,y,z) we have

$$\operatorname{rank}\begin{bmatrix}\nabla f(x,y,z)\\ \nabla g_1(x,y,z)\\ \nabla g_2(x,y,z)\end{bmatrix}=2, \iff \det\begin{bmatrix} y & x & 2z \\ 2x & 2y & 0 \\ 2x & 2y & 2z\end{bmatrix}=0.$$

Computing the determinant by third column,

$$2z(2y^2-2x^2)=0, \iff z(y-x)(y+x)=0.$$

Candidates are therefore the points $(x,y,0)$, $x,y\in\mathbb{R}$, (x,x,z), $(x,-x,z)$, with $x,z\in\mathbb{R}$. Now

$$(x,y,0)\in\mathscr{M}, \iff \begin{cases} x^2=1, \\ \\ x^2+y^2=1, \end{cases} \iff (x,y,0)=(\pm1,0,0).$$

Similarly

$$(x,x,z)\in\mathscr{M}, \iff \begin{cases} x^2+z^2=1, \\ \\ 2x^2=1, \end{cases} \iff \left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right);$$

$$(x,-x,z)\in\mathscr{M}, \iff \begin{cases} -x^2+z^2=1, \\ \\ 2x^2=1, \end{cases} \iff \left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}},\pm\frac{\sqrt{3}}{\sqrt{2}}\right);$$

It is easy to conclude: $(\pm1,0,0)$ are the points at min distance, $\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}},\pm\frac{\sqrt{3}}{\sqrt{2}}\right)$ are at max distance. □

2.6. Exercises

Exercise 2.6.1. Compute the following directional derivatives:

i) $D_{(\sqrt{3},1)}f(1,1)$ where $f(x,y):=\log(1+xy)$.
ii) $D_{(2,2)}f(1,0)$ where $f(x,y):=\arctan(x+y)$.
iii) $D_{(1,1)}f(0,0)$ where $f(x,y):=\frac{x^2y}{|x|+y^2}$ for $(x,y)\neq\vec{0}$ and $f(0,0)=0$.
iv) $D_{(-1,1)}f(0,0)$ where $f(x,y)=\frac{xy}{x^2+y^4}$, for $(x,y)\neq\vec{0}$ and $f(0,0)=0$.
v) $D_{(-1,-2)}f(0,0)$, where $f(x,y):=\frac{y(e^x-1)}{\sqrt{x^2+y^2}}$ for $(x,y)\neq\vec{0}$ and $f(0,0)=0$.

Exercise 2.6.2. For each of the following functions say whether f is continuous at point $(0,0)$, there exist $\partial_x f(0,0)$, $\partial_y f(0,0)$, and f is differentiable in $(0,0)$.

1. $f(x,y):=\begin{cases} \frac{x^3}{x^2+y^2}, & (x,y)\neq 0_2, \\ \\ 0, & (x,y)=0_2. \end{cases}$

2. $f(x,y):=\begin{cases} \frac{x^2+y^2}{|x|+|y|}, & (x,y)\neq 0_2, \\ \\ 0, & (x,y)=0_2. \end{cases}$

3. $f(x,y):=\begin{cases} \frac{x^2y^3}{(x^2+y^2)^2}, & (x,y)\neq 0_2, \\ \\ 0, & (x,y)=0_2. \end{cases}$

4. $f(x,y):=\begin{cases} \frac{x^2y}{x^2+y^2}+x-y, & (x,y)\neq 0_2, \\ \\ 0, & (x,y)=0_2. \end{cases}$

Exercise 2.6.3. Show that the function $f(x,y)=x\sqrt{x^2+y^2}$, $(x,y)\in\mathbb{R}^2$ is differentiable on $\mathbb{R}^2$.

Exercise 2.6.4. Determine the stationary points of each of the following functions:

i) $f(x,y):=xy(x+1)$.
ii) $f(x,y):=x^2+y^2+xy$.
iii) $f(x,y):=x^3+y^3+2x^2+2y^2+x+y$.
iv) $f(x,y:)=xe^y+ye^x$.
v) $f(x,y,z):=(x^3-3x-y^2)z^2+z^3$.

EXERCISE 2.6.5. Find the value of the parameter $\lambda \in \mathbb{R}$ such that the function $f(x,y) := x^2 + \lambda y^2 - 4x + 2y$ has a stationary point in $(2,-1)$. What kind of point is this?

EXERCISE 2.6.6. Let$f(x,y) := x^2(1-y)$ on $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + |y| \leqslant 4\}$. Study the sign of f, determine its eventual stationary points on D and min/max of f on D. Determine $f(D)$.

EXERCISE 2.6.7. Discuss min/max of $f(x,y) := xye^{-xy}$ on $D = \{(x,y) \in \mathbb{R}^2 \ : \ 1 \leqslant x \leqslant 4, \ y \geqslant 0, \ |xy| \leqslant 1\}$.

EXERCISE 2.6.8. For each of the following functions a) find the stationary points, b) find any min/max in the domain, c) find the image of the domain.

i) $f(x,y) = x^4 + y^4 - xy$, on $D = \mathbb{R}^2$.
ii) $f(x,y) = x\left((\log x)^2 + y^2\right)$ on $D =]0,+\infty[\times\mathbb{R}$.
iii) $f(x,y) = xy(x+y)$, on $D = \mathbb{R}^2$.
iv) $f(x,y,z) = x^2 + y^2 + z^2 - 2xy + 2xz$ on $D = \mathbb{R}^3$.
v) $f(x,y,z) = x^4 + y^4 + z^4 - xyz$, on $D = \mathbb{R}^3$.

EXERCISE 2.6.9. Let

$$f(x,y) = (x^2+y^2)^2 - 3x^2y, \ (x,y) \in \mathbb{R}^2.$$

i) Determine $\lim_{(x,y)\to\infty_2} f(x,y)$ (if any). ii) Find stationary points of f. iii) Find eventual global min/max of f on $\mathbb{R}^2$ and find $f(\mathbb{R}^2)$.

EXERCISE 2.6.10. Let

$$f(x,y,z) := (x^2+y^2+z^2)^2 - xyz, \ (x,y,z) \in \mathbb{R}^3.$$

i) Show that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. ii) Find stationary points of f. iii) Show that f has global minimum on $\mathbb{R}^3$ and find $f(\mathbb{R}^3)$.

EXERCISE 2.6.11. Let $f(x,y) := x^2\left(y^2 - (x-1)^2\right)$, $(x,y) \in \mathbb{R}^2$. i) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? If yes, compute it. ii) Find and classify the stationary points of f on $\mathbb{R}^2$. iii) What about extrema of f on $\mathbb{R}^2$? Determine $f(\mathbb{R}^2)$. iv) Show that f has min/max on $D := \{(x,y) \in \mathbb{R}^2 \ : \ y \leqslant 0, \ 0 \leqslant x \leqslant y+1\}$ and find them. What is $f(D)$?

EXERCISE 2.6.12. Consider the function $f(x,y) := x^4 + y^4 - 8(x^2+y^2)$ on $\mathbb{R}^2$. i) Compute $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find and classify the stationary points of f. What can you say about global min/max points of f? What about $f(\mathbb{R}^2)$? iii) Find the min/max points of f on the domain $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leqslant 9\}$.

EXERCISE 2.6.13. Consider the function

$$f(x,y) := \begin{cases} \frac{x^5y^2}{(x^4+y^2)^2}, & (x,y) \neq 0_2, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

i) Say if f is continuos, differentiable in 0_2 (and in this case compute $\nabla f(0_2)$). ii) Find any stationary points of f on $\mathbb{R}^2$ and discuss their nature. Does f has min/max points on $\mathbb{R}^2$? iii) Show that f has min/max on the domain $D = \{(x,y) \in \mathbb{R}^2 \ : \ |x| \leqslant 1, \ |y| \leqslant 1\}$ and find them.

EXERCISE 2.6.14. Let f be the function defined as

$$f(x,y) := \begin{cases} xye^{\frac{xy}{x^2+y^2}}, & (x,y) \in \mathbb{R}^2\backslash\{0_2\}, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

i) Say if f is continuous and differentiable at 0_2. ii) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? (in the case affirmative, what is the value?). Is f bounded on $\mathbb{R}^2$? iii) Show that f has min/max on $\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leqslant 1\}$ and find them.

EXERCISE 2.6.15. Determine min/max of f on the set D in the following cases:

i) $f = x + y$, $D = \{(x, y) \ : \ x^2 + y^2 = 1\}$;
ii) $f = 2x^2 + y^2 - x$, $D = \{(x, y) \ : \ x^2 + y^2 = 1\}$;
iii) $f = xy$, $D = \{(x, y) \ : \ x^2 + y^2 + xy - 1 = 0\}$;
iv) $f = x^2 + 5y^2 - \frac{1}{2}xy$, $D = \{(x, y) \ : \ x^2 + 4y^2 = 4\}$.

EXERCISE 2.6.16. Determine min/max of f on the set D in the following cases:

i) $f = x - 2y + 2z$, $D = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 9\}$;
ii) $f = z^2 e^{xy}$, $D = \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\}$.

EXERCISE 2.6.17. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 + 1, \ z = 2x^2 + y^2\}$. Show that i) $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$, ii) $\mathscr{M}$ is compact. iii) $\mathscr{M}$ has points of maximum quote: find them.

EXERCISE 2.6.18. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ z^2 = xy + 1\}$. Show that i) $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$,ii) $\mathscr{M}$ is not compact. iii) Show that there exists points of $\mathscr{M}$ at minimum distance to the origin and find them.

EXERCISE 2.6.19. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2 + z^2)^2 - xyz = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Say if $\mathscr{M}$ is compact or not. iii) Determine, if they exists, points on $\mathscr{M}$ at maximum distance to the origin.

EXERCISE 2.6.20. Let $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 0, \ x^2 - y^2 = 1\right\}$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Say if $\mathscr{M}$ is compact or less. iii) Noticed that 0_3 is not on $\mathscr{M}$, show that exists points of $\mathscr{M}$ at minimum distance from 0_3 and find them.

EXERCISE 2.6.21. Let $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - xy + y^2 - z^2 = 1, \ x^2 + y^2 = 1\right\}$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Show that $\mathscr{M}$ is compact. iii) Find stationary points of $f(x, y, z) = xyz$ on $\mathscr{M}$. What can you say about the problem to find extrema of f on $\mathscr{M}$?

EXERCISE 2.6.22. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Is $\mathscr{M}$ compact? iii) Find points of $\mathscr{M}$ at minimum distance from the origin 0_3.

EXERCISE 2.6.23. Find the stationary points of $f(x, y, z) := xyz$, $(x, y, z) \in \mathbb{R}^3$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (here $a, b, c > 0$). Deduce min/max of f on the ellipsoid.

EXERCISE 2.6.24. Determine min/max of $f(x, y, z) = xy + yz + zx$ on the plane $x + y + z = 3$.

EXERCISE 2.6.25. Compute the min/max distance of the point $(0, 1, 0)$ to the following subset of $\mathbb{R}^3$:

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ \\ x^2 + y^2 = x. \end{cases}$$

EXERCISE 2.6.26. Consider the set $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z = x^2 + y^2, \ x + y + z = 0\right\}$. Show that $\mathscr{M}$ is not empty and is a . . . Find points of $\mathscr{M}$ with min/max quote.

EXERCISE 2.6.27. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1, \ 2z - 3x = 0\}$ and $f(x, y, z) := xz$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Show that $\mathscr{M}$ is compact. iii) Find extrema of f on $\mathscr{M}$.

EXERCISE 2.6.28. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ : \ 2x^2 + 2y^2 - z^2 = 1, \ (x - y)^2 + z = 2\}$. i) Show that $\mathscr{M} \neq \emptyset$ is the zero set of a submersion on $\mathscr{M}$. ii) Show that $\mathscr{M}$ is not compact. iii) Find stationary points of $f(x, y, z) := z$ on $\mathscr{M}$.

EXERCISE 2.6.29. Let
$$f(x,y,z) := \frac{\sqrt{x^2+\frac{y^2}{4}}-3}{4} + z^2,\ (x,y,z)\in\mathbb{R}^3.$$
i) Compute $\lim_{(x,y,z)\to\infty_3} f(x,y,z)$: what can you deduce by this about min/max f? ii) Find and classify all the stationary points of f on $\mathbb{R}^3$. Find, if there exist, min/max of f on $\mathbb{R}^3$. What is $f(\mathbb{R}^3)$? iii) Let $\mathscr{M} := \{(x,y,z)\in\mathbb{R}^3 \ :\ f(x,y,z)=1\}$. Prove that $\mathscr{M}\neq\emptyset$ is the zero set of a submersion on $\mathscr{M}$. Is $\mathscr{M}$ compact? iv) Show that there exists points of $\mathscr{M}$ at min/max distance to the origin. Find them.

EXERCISE 2.6.30. Find
$$\max\{xy^2z^3 \ :\ x,y,z>0,\ x+y+z=6\}.$$

CHAPTER 3

Vector fields

Consider a function $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ differentiable on D. Then

$$\nabla f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

A function $\vec{F}$ of this type, that is a function $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$, is called *vector field*. Vector fields are very important entities in Physics. The classical example is a *force field*. For example, the *gravitational field* induced by a point mass m positioned at point $\vec{x}_0 \in \mathbb{R}^3$ is

$$\vec{F}(\vec{x}) = -Gm\frac{\vec{x}-\vec{x}_0}{\|\vec{x}-\vec{x}_0\|^3}, \quad \vec{x} \in \mathbb{R}^3 \backslash \{\vec{x}_0\} =: D,$$

where G is the *universal gravitational constant*.

An important concept related to a vector field is its *potential*, namely a function f, if any, such that $\nabla f = \vec{F}$. For example, in the case of the gravitational field it is easy to check that

$$f(\vec{x}) = Gm\frac{1}{\|\vec{x}-\vec{x}_0\|}, \ \vec{x} \in D,$$

is a potential for $\vec{F}$ (check this). The main scope of this Chapter is to understand how the problem of determining a potential of a vector field can be solved in general.

In dimension $d = 1$, this problem is well known and it consists in finding a *primitive* for function of one real variable: given F, determine f such that $f' = F$. By the Fundamental Theorem of Integral Calculus, we know that if $f \in \mathscr{C}([a,b])$ then the solution always exists and it is given by

$$f(x) = \int_c^x F(y)\, dy.$$

As we will see, the multidimensional version we're going to study in this Chapter, is much more involved and even extremely regular fields do not have any potential. We will discover what is behind this and we will recover that, under suitable definition of integral, a formula very similar to the previous one holds true.

Chapter requirements: differential calculus for functions of several variables, integration for function of one real variable, primitives.

3.1. Preliminaries

We start with the

Definition 3.1.1

A continuous function $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ on D open set is called **vector field on** D.

Practical examples of vector fields are, as in the introduction, *force fields* and *velocity fields*. These last are used to describe, for instance, velocities in a fluid or gas. Imagine a fluid in movement: at every point $\vec{x}$ the molecule of water at x has ha velocity $\vec{v}(vecx)$. If we consider a fluid in space this function is a vector field.

Apart for the case $d = 1$, even for $d = 2$ it is not immediate to visualize a vector field. Physics suggests an interesting way: in practice, at every point $\vec{x} \in D$ we may trace a vector $\vec{F}(\vec{x})$.

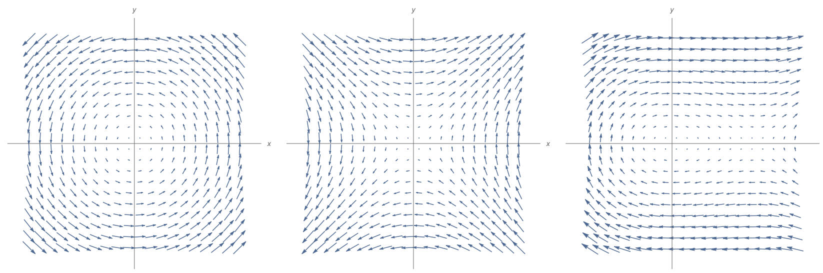


FIGURE 1. Left to right: vector fields $(-y, x)$, (x, y) and $(y, x^2 - x)$.

Definition 3.1.2

We say that $\vec{F}$ is **conservative on** D if there exists $f \in \mathscr{C}^1(D)$ such that $\vec{F} = \nabla f$ on D. The function f is called **potential** of $\vec{F}$.

If $\vec{F} = (f_1, \dots, f_d)$ then

$$\nabla f = \vec{F}, \text{ on } D, \iff \begin{cases} \partial_1 f(\vec{x}) = f_1(\vec{x}), \\ \vdots \\ \partial_d f(\vec{x}) = f_d(\vec{x}), \end{cases} \quad \forall \vec{x} \in D.$$

Clearly, if f is a potential for $\vec{F}$, also $f + c$, where c is a real constant, is a potential for $\vec{F}$ because $\nabla(f + c) = \nabla f + \nabla c = \vec{F} + \vec{0} = \vec{F}$. In dimension 1, because potentials are the primitives, if the domain is an interval all the potentials differ by an additive constant. In higher dimension this remains true if the domain D is made of one piece, that is if it is connected.

Proposition 3.1.3

Let D be a connected set and f, g potentials of the vector field $\vec{F} \in \mathscr{C}(D)$. Then $f - g \equiv$constant.

PROOF. Assume $\nabla f = \vec{F} = \nabla g$. Then $\nabla(f - g) = \vec{0}$. Let $h := f - g$, so $\nabla h = \vec{0}$. The conclusion follows from the

Lemma 3.1.4

If $\nabla h = \vec{0}$ on D connected then h is constant.

PROOF. (Lemma) Pick two points $\vec{x}, \vec{y} \in D$. We show $h(\vec{x}) = h(\vec{y})$. Since D is connected, there exists a curve $\vec{\gamma} = \vec{\gamma}(t)$ in D joining $\vec{x}$ to $\vec{y}$, that is $\vec{\gamma} : [a, b] \longrightarrow \mathbb{R}^d$ such that $\vec{\gamma}(a) = \vec{x}$, $\vec{\gamma}(b) = \vec{y}$. Consider $\phi(t) := h(\vec{\gamma}(t))$. We may assume $\vec{\gamma}$ is regular, that is $\exists \vec{\gamma}'(t)$ for every $t \in [a, b]$. Since $\vec{\gamma}(t) \in D$ for every t, we have

$$\phi'(t) = \nabla h(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) = 0, \ \forall t \in [a, b], \implies \phi(t) \equiv C,$$

that is, in particular, $\phi(a) = \phi(b)$. But $\phi(a) = h(\vec{\gamma}(a)) = h(\vec{x})$ and $\phi(b) = h(\vec{\gamma}(b)) = h(\vec{y})$. □

EXAMPLE 3.1.5. *Find the potentials for the field*

$$\vec{F}(x, y) = (y, x), \quad (x, y) \in \mathbb{R}^2 =: D.$$

SOL. — We have to find $f \in \mathscr{C}^1(\mathbb{R}^2)$ such that

$$\begin{cases} \partial_x f(x, y) = y, \\ \\ \partial_y f(x, y) = x, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

Consider the first equation, $\partial_x f(x, y) = y$. We can see this as a problem of one variable primitive and say that

$$f(x, y) = \int y \, dx = yx + c,$$

where c is a free constant: of course constant w.r.t. x. Then we may imagine $c = c(y)$, that is

$$f(x, y) = xy + c(y).$$

To find c we use the second equation, $\partial_y f(x, y) = x$. Indeed

$$\partial_y f(x, y) = x, \iff x + c'(y) = x, \iff c'(y) = 0, \iff c(y) \equiv c,$$

and we deduce $f(x, y) = xy + c$, $c \in \mathbb{R}$. □

3.2. Irrotational fields

As reminded, in dimension 1 the problem to find a primitive of a function F has always an answer if F is continuous. Moving to dimensions $\geqslant 2$, things are different: even if $\vec{F}$ is continuous, the problem $\nabla f = \vec{F}$ might not have any solution!

EXAMPLE 3.2.1. *Show that the field*

$$\vec{F}(x, y) = (y, -x), \quad (x, y) \in \mathbb{R}^2 =: D,$$

has not any potential.

SOL. — We have to find $f \in \mathscr{C}^1(\mathbb{R}^2)$ such that

$$\begin{cases} \partial_x f(x, y) = y, \\ \\ \partial_y f(x, y) = -x, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

Consider the first equation, $\partial_x f(x, y) = y$. We can see this as a problem of one variable primitive and say that

$$f(x, y) = \int y \, dx = xy + c(y).$$

Now, by the second equation

$$\partial_y f(x, y) = -x, \iff x + c'(y) = -x, \iff c'(y) = -2x.$$

Now this is impossible because c must be constant in x! We deduce that it is impossible that f exists. □

In the previous Example actually $\vec{F} \in \mathscr{C}^\infty$ (its components are polynomials). The reason why this is not sufficient to ensure the existence of a potential has to be found in a consequence of the following

Theorem 3.2.2: Schwarz

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be such that $\partial_i f \in \mathscr{C}^1(D)$ for every $i = 1, \dots, d$ (that is $\partial_j(\partial_f) \in \mathscr{C}(D)$ for every $i, j = 1, \dots, d$). Then
$$\partial_j(\partial_i f) = \partial_i(\partial_j f), \text{ on } D.$$

The proof is technical and not interesting here. Schwarz's flipping rule (obvious if $i = j$) says that we can always flip the order of derivatives in a nested derivation provided all the partial derivatives are continuous.

Example 3.2.3. *Check the Schwarz rule on a simple case: $f(x, y) = xy^2 + xy$.*

Sol. — Clearly $\partial_x f = y^2 + y$ and $\partial_y f = 2xy + x$ are both continuous. Furthermore
$$\partial_x(\partial_y f) = \partial_x(2xy + x) = 2y + 1,$$
$$\partial_y(\partial_x f) = \partial_y(y^2 + y) = 2y + 1.$$
As you can see $\partial_x(\partial_y f) \equiv \partial_y(\partial_x f)$. □

Schwarz's flipping rule has an immediate consequence:

Proposition 3.2.4

Let $\vec{F} \in \mathscr{C}^1(D)$, $\vec{F} = (f_1, \dots, f_d)$ be a conservative vector field. Then

(3.2.1) $$\partial_i f_j \equiv \partial_j f_i, \text{ on } D, \ \forall i, j = 1, \dots, n.$$

Proof. Since $\vec{F} = \nabla f$ for some $f \in \mathscr{C}^1(D)$, that is $f_i = \partial_i f$, and $\vec{F} \in \mathscr{C}^1$, it follows that $\partial_i f \in \mathscr{C}^1$, $i = 1, \dots, d$. Therefore, by Schwarz's theorem
$$\partial_j f_i = \partial_j \partial_i f \overset{Schwarz}{=} \partial_i \partial_j f = \partial_i f_j. \quad \square$$

Definition 3.2.5

A $\mathscr{C}^1(D)$ vector field $\vec{F}$ fulfilling (3.2.1) is called **irrotational vector field**.

Example 3.2.6. It is easy to check that in the Examples 3.2.1 the proposed fields are not irrotational. For instance,
$$(y, -x) \text{ is irrotational} \iff \partial_y(y) = \partial_x(-x), \iff 1 = -1,$$
which is false. □

Therefore, to be conservative the vector field $\vec{F} \in \mathscr{C}^1$ must be first of all irrotational. The natural question is if to be irrotational is a sufficient condition to be conservative? The answer is no!

EXAMPLE 3.2.7 (Important!). *The field*

$$\vec{F}(x,y) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad (x,y) \in \mathbb{R}^2 \backslash \{0_2\}$$

is irrotational but not conservative on $\mathbb{R}^2\backslash\{0_2\}$.

SOL. — Let us check first that $\vec{F}$ is irrotational. It is evident that $\vec{F} \in \mathscr{C}^1$ and $\vec{F}$ is irrotational iff

$$\partial_y \left(-\frac{y}{x^2+y^2}\right) \equiv \partial_x \left(\frac{x}{x^2+y^2}\right)$$

We have

$$\partial_y \left(-\frac{y}{x^2+y^2}\right) = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{x^2+y^2}, \quad \partial_x \left(\frac{x}{x^2+y^2}\right) = -\frac{x^2-y^2}{x^2+y^2}.$$

Therefore $\vec{F}$ is irrotational. Let us assume that a potential f exists. Then

$$\partial_x f(x,y) = -\frac{y}{x^2+y^2}, \iff f(x,y) = \int -\frac{y}{x^2+y^2}\, dx + c(y) = -\frac{1}{y}\int \frac{1}{1+\left(\frac{x}{y}\right)^2}\, dx + c(y) = -\arctan\frac{x}{y} + c(y).$$

This if $y \neq 0$. If $y = 0$,

$$\partial_x f(x,0) = 0, \iff f(x,0) = c(0),$$

Therefore the candidate is

$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + c(y), & y \neq 0, \\ \\ c(0), & y = 0. \end{cases}$$

On the other hand, if $y \neq 0$,

$$\partial_y f(x,y) = \frac{x}{x^2+y^2}, \iff \partial_y \left(-\arctan\frac{x}{y} + c(y)\right) = \frac{x}{x^2+y^2}, \iff c'(y) \equiv 0,$$

so $c \equiv C$. We derive then that

$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + C, & y \neq 0, \\ \\ C, & y = 0. \end{cases}$$

We are done apparently. But... looking carefully to f we see that the f we found is not even continuous! To see this consider a point $(x,0)$ with $x > 0$. If $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0+$ then

$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(+\infty) + C = -\frac{\pi}{2} + C.$$

But if $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0-$ we have

$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(-\infty) + C = +\frac{\pi}{2} + C.$$

The conclusion is that $\lim_{(x,y)\longrightarrow(x,0)} f(x,y)$ doesn't exists, for any $x > 0$. $\square$

3.3. Path Integral

An important concept of Physics is the *work done by a force along a path*. This quantifies the energy spent by a force to move a mass from a point to another along a certain path. We may describe a path through a curve $\vec{\gamma} \in \mathscr{C}^1([a,b];\mathbb{R}^d)$. At point $\vec{\gamma}(t)$, force $\vec{F}(\vec{\gamma}(t))$ acts on the mass. The component of $\vec{F}(\vec{\gamma}(t))$ along $\vec{\gamma}$ at is measured by

$$\|\vec{F}(\gamma(t))\|\|\gamma'(t)\|\cos\theta(t),$$

where $\theta(t)$ is the angle made by $\vec{F}(\vec{\gamma}(t))$ and $\vec{\gamma}'(t)$. Previous formula is nothing but the scalar product

$$\vec{F}(\gamma(t))\cdot\gamma'(t),$$

where $\cdot$ is the standard scalar product of $\mathbb{R}^d$. This is a static picture. Considering now a time frame $[a,b]$, the natural operation is to consider

$$\int_a^b \vec{F}(\gamma(t))\cdot\gamma'(t)\,dt.$$

Definition 3.3.1

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a continuous vector field on D, $\vec{\gamma} \in \mathscr{C}^1([a,b];D)$ a continuous curve in D. We call **path integral of $\vec{F}$ along** $\vec{\gamma}$ the integral

$$\int_{\vec{\gamma}} \vec{F} := \int_a^b \vec{F}(\vec{\gamma}(t))\cdot\vec{\gamma}'(t)\,dt.$$

If $\vec{\gamma}$ is a **circuit**, that is $\vec{\gamma}(a)=\vec{\gamma}(b)$, we call **circulation** of $\vec{F}$ along $\vec{\gamma}$ the integral

$$\oint_{\vec{\gamma}} \vec{F} := \int_{\vec{\gamma}} \vec{F}.$$

The following result is the analogous of Fundamental Theorem of Integral Calculus for line integrals:

Proposition 3.3.2

Let $\vec{F} \in \mathscr{C}(D)$ be a conservative vector field with potential f. Then, if $\vec{\gamma} \in \mathscr{C}^1([a,b];D)$

$$\int_{\vec{\gamma}} \vec{F} = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)). \tag{3.3.1}$$

In particular, if $\vec{\gamma}$ is a **circuit** in D,

$$\oint_{\vec{\gamma}} \vec{F} = 0. \tag{3.3.2}$$

Proof. If $\vec{F} = \nabla f$ then, by the fundamental formula of integral calculus,

$$\int_{\vec{\gamma}} \vec{F} = \int_a^b \nabla f(\gamma(t))\cdot\gamma'(t)\,dt = \int_a^b \frac{d}{dt}f(\vec{\gamma}(t))\,dt = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)),$$

because of the fundamental thm of integral calculus. The (3.3.3) follows immediately because for a circuit $\vec{\gamma}$ we have $\vec{\gamma}(b)=\vec{\gamma}(a)$. $\square$

Example 3.3.3. *Compute $\oint_{x^2+y^2=1} \vec{F}$ for the $\vec{F}$ of Example 3.2.7.*

Sol. — Let $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. We may parametrize the unit circle as $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\oint_\gamma \vec{F} = \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t) + \frac{\cos t}{1}\cos t\, dt = \int_0^{2\pi} dt = 2\pi. \qquad \square$$

We have seen that (3.3.3) is a necessary condition for $\vec{F}$ to be conservative. It turns out that if D is connected ("made of one piece"), then it is also sufficient:

Theorem 3.3.4: fundamental theorem of calculus for fields

Let $\vec{F} \in \mathscr{C}(D)$ be a vector field on $D \subset \mathbb{R}^d$ open and connected and such that

$$\oint_\gamma \vec{F} = 0, \quad \forall \gamma \in \mathscr{C}^1([a,b]; D), \text{ circuit.} \tag{3.3.3}$$

Then $\vec{F}$ is conservative and all the possible potentials are

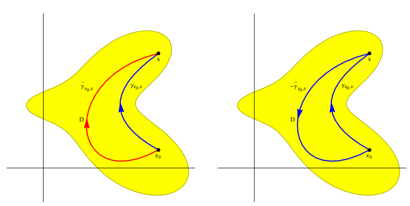
$$f(\vec{x}) = \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} + c,\ \vec{x} \in D,\ c \in \mathbb{R}, \tag{3.3.4}$$

where $\gamma_{\vec{x}_0,\vec{x}}$ is any regular path contained in D that joins $\vec{x}_0$ to $\vec{x}$.

Proof. We will prove that i) formula (3.3.4) is well posed, that is, the line integral is independent of the specific path connecting $\vec{x}_0$ to $\vec{x}$; ii) $\nabla f = \vec{F}$.

First: Let $\widetilde{\gamma}_{\vec{x}_0,\vec{x}}$ be a second path connecting $\vec{x}_0$ to $\vec{x}$. We want to prove that

$$\int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} = \int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F}.$$



Let $-\widetilde{\gamma}_{\vec{x}_0,\vec{x}}$ be the path $\widetilde{\gamma_{\vec{x}_0,\vec{x}}}$ oriented in the opposite direction. Formally, if

$$\widetilde{\gamma_{\vec{x}_0,\vec{x}}} : [a,b] \longrightarrow \mathbb{R}^d, \quad -\widetilde{\gamma_{\vec{x}_0,\vec{x}}}(t) := \widetilde{\gamma_{\vec{x}_0,\vec{x}}}(a+b-t).$$

Then, if we consider the path formed made first by $\gamma_{x_0,x}$ and then by $-\widetilde{\gamma}_{x_0,x}$ we get a circuit that we will denote by the symbol $\gamma_{\vec{x}_0,\vec{x}} - \widetilde{\gamma}_{\vec{x}_0,\vec{x}}$. Then, by our assumption

$$0 = \oint_{\gamma_{\vec{x}_0,\vec{x}} - \widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F} = \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} + \int_{-\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F}.$$

A straightforward calculation shows that

$$\int_{-\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F} = -\int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F}.$$

Therefore

$$\int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} = \int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F}.$$

This proves the well posedness of definition (3.3.4). The second step consists in proving that $\nabla f = \vec{F}$. We have to prove that

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \vec{F}(\vec{x}) \cdot \vec{h} + o(\vec{h}).$$

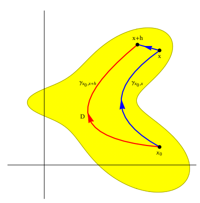
First,

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{\gamma_{\vec{x}_0,\vec{x}+\vec{h}}} \vec{F} - \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F}.$$

Since the integral does not depend on the particular path connecting $\vec{x}_0$ to $\vec{x}+\vec{h}$, we can write

$$\int_{\gamma_{\vec{x}_0,\vec{x}+\vec{h}}} \vec{F} = \int_{\gamma_{\vec{x}_0,\vec{x}}+[\vec{x},\vec{x}+\vec{h}]} \vec{F} = \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} + \int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F},$$

where $[\vec{x},\vec{x}+\vec{h}]$ stands for the linear segment joining $\vec{x}$ to $\vec{x}+\vec{h}$.



Therefore

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F}.$$

Now, the natural parametrization of the segment $[\vec{x},\vec{x}+\vec{h}]$ is $\gamma(t) = \vec{x}+t\vec{h}$, $t \in [0,1]$ so $\gamma'(t) = \vec{h}$. Hence

$$\begin{aligned} \int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F} &= \int_0^1 \vec{F}(\vec{x}+t\vec{h}) \cdot \vec{h}\, dt = \int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x}) + \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt \\ &= \vec{F}(\vec{x}) \cdot \vec{h} + \underbrace{\int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt.}_{\varepsilon(\vec{h})}. \end{aligned}$$

It remains to prove that $\varepsilon(\vec{h}) = o(\vec{h})$. Notice first that Now,

$$|\varepsilon(\vec{h})| = \left|\int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt\right| \overset{\triangle}{\leqslant} \int_0^1 \left\|\left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\right\| dt \overset{CS}{\leqslant} \left(\int_0^1 \left\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right\| dt\right) \|\vec{h}\|,$$

thus

$$\frac{|\varepsilon(\vec{h})|}{\|\vec{h}\|} \leqslant \int_0^1 \left\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right\| dt.$$

Since $\vec{F}$ is continuous, $\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\| \longrightarrow 0$ when $\vec{h} \longrightarrow \vec{0}$ and by this (with some work to be done) the conclusion follows. $\square$

In general, to check null circulations condition (3.3.3) can be a prohibitive task. Under certain conditions on the domain, however, an irrotational field verifies condition (3.3.3):

Proposition 3.3.5

Let $\vec{F} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ an **irrotational** vector field. Assume that D verifies the following condition:

$$\forall\, \gamma \text{ circuit, } \exists \Omega \subset D \text{ open } : \gamma = \partial\Omega.$$

Then, $\vec{F}$ is conservative.

The proof of this proposition is a consequence of the *Green formula*, a formula involving multiple integration. We will postpone the proof in Section 4.6.

3.4. Exercises

Exercise 3.4.1. Compute $\int_\gamma \vec{F}$ in the following cases:

(1) $\vec{F}(x,y) := (y^3 + x, -\sqrt{x})$ on $D = [0,+\infty[\times\mathbb{R}$, γ of equation $x = y^2$ connecting $(0,0)$ to $(1,1)$.
(2) $\vec{F}(x,y) := (y^2, 2xy+1)$ on $D = \mathbb{R}^2$, γ of equation $y = \sqrt{|x-1|}$, $x \in [0,2]$.
(3) $\vec{F}(x,y) := (\sqrt{y}, x^3 + y)$ on $D = \mathbb{R} \times [0,+\infty[$ along $y = x^2$ connecting $(1,1)$ to $(2,4)$.
(4) $\vec{F}(x,y) := \left(\frac{x+1}{y-1}, \frac{y+1}{x-1}\right)$ on $D = \{(x,y) \in \mathbb{R}^2 \ : \ y \neq 1,\ x \neq 1\}$, along the segment connecting $(0,0)$ to $(1/2, 1/2)$.
(5) $\vec{F}(x,y) := (\log(1+y), \log(1+x))$, on $D =]-1,+\infty[^2$, along the segment connecting $(1,0)$ to $(0,1)$.
(6) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$ on $D = \mathbb{R}^3$ along the helix $\gamma(t) = (r\cos t, r\sin t, kt)$, $t \in [0,2\pi]$.

Exercise 3.4.2. For each of the following vector fields on the given domains, check if they are irrotational, conservative and, in this case, find a potential:

(1) $\vec{F}(x,y) := (x, y-1)$, $(x,y) \in \mathbb{R}^2$;
(2) $\vec{F}(x,y) := (y, x)$, $(x,y) \in \mathbb{R}^2$;
(3) $\vec{F}(x,y) := (x, -y)$, $(x,y) \in \mathbb{R}^2$;
(4) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$, $(x,y,z) \in \mathbb{R}^3$;

Exercise 3.4.3. Find al possible values for $a, b, c \in \mathbb{R}$ such that the field

$$\vec{F}(x,y) := \left(ax^3 + by + 3x^2y^2, cx^4 + 2x^3y + 1\right),$$

be irrotational on $\mathbb{R}^2$. In the case say if it is also conservative and find all the potentials.

Exercise 3.4.4. Consider the vector field

$$\vec{F}(x,y) := \left(\sin\frac{x}{y} + \frac{x}{y}\cos\frac{x}{y}, -\frac{x^2}{y^2}\cos\frac{x}{y} + 3\right), \text{ on } D = \mathbb{R}\times]0,+\infty.$$

Check that $\vec{F}$ is irrotational and say if it is also conservative. In this case compute a potential f such that $f(\pi, 1) = 3$.

Exercise 3.4.5. Consider the vector field

$$\vec{F}(x,y) := \left(\frac{y}{x^2}\cos\frac{y}{x}, -\frac{a}{x}\cos\frac{y}{x}\right), \text{ on } D =]0,+\infty[\times\mathbb{R}.$$

Find values of $a \in \mathbb{R}$ such that $\vec{F}$ is irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

EXERCISE 3.4.6. Consider the vector field

$$\vec{F}(x,y) := \left(\frac{1}{1+y^2}, -\frac{2xy}{(1+y^2)^2}\right),\ (x,y)\in\mathbb{R}^2.$$

Is $\vec{F}$ irrotational? Conservative? Compute, the path integral $\int_\gamma \vec{F}$ where $\gamma(t) = \left(e^{\sin t}, \frac{2\cos t}{1+(\cos t)^2}\right)$, $t\in[0,\pi]$.

EXERCISE 3.4.7. Consider the vector field

$$\vec{F}(x,y) := \left(-\frac{axy}{(x^2+y^2)^2}, \frac{bx^2-y^2}{(x^2+y^2)^2}\right),\ \text{on } D=\mathbb{R}^2\backslash\{0_2\}.$$

Find values of $a,b\in\mathbb{R}$ such that $\vec{F}$ be irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

EXERCISE 3.4.8. Consider the vector field

$$\vec{F}(x,y,z) := \left(a(x,y,z), x^2+2yz, y^2-z^2\right),\ (x,y,z)\in\mathbb{R}^3,$$

where a is a $\mathscr{C}^1$ function. Find all the possible a in such a way that $\vec{F}$ be irrotational. Show that there is a unique a null as $y=z=0$. In that case find all the potentials of $\vec{F}$.

EXERCISE 3.4.9. Find $a,b,c,d\in\mathbb{R}$ in such a way that the vector field

$$\vec{F}(x,y) := \left(\frac{ax+by}{x^2+y^2}, \frac{cx+dy}{x^2+y^2}\right),\ (x,y)\in\mathbb{R}^2\backslash\{0_2\}$$

be irrotational. For such values, find those such that $\vec{F}$ is conservative and find also its potentials.

EXERCISE 3.4.10. Let g be the vector field defined by

$$\vec{F}(x,y) := \left(\frac{ax^2+by^2}{(x^2+y^2)^2}, \frac{cxy}{(x^2+y^2)^2}\right),\ \ (x,y)\in D:=\mathbb{R}^2\backslash\{0_2\},\ \ (a,b,c\in\mathbb{R}).$$

i) Find $a,b,c\in\mathbb{R}$ such that $\vec{F}$ is irrotational. ii) Find $a,b,c\in\mathbb{R}$ such that $\vec{F}$ is conservative: for such a,b,c find the potentials of $\vec{F}$ (hint: . start with $\partial_y f = f_2(x,y)\dots$)

EXERCISE 3.4.11. Let $\vec{F}$ be the vector field defined as

$$\vec{F}(x,y) := \left(\frac{axy^2}{(x^2+y^2)^{1/2}}, \frac{bx^2y+cy^3}{(x^2+y^2)^{1/2}}\right),\ \ (x,y)\in\mathbb{R}^2\backslash\{0_2\} =: D.$$

i) Find $a,b,c\in\mathbb{R}$ such that $\vec{F}$ be irrotational. ii) For the values found at i), say if $\vec{F}$ is conservative on $\mathbb{R}^2\backslash\{(0,y)\ :\ y\geqslant 0\}$ and on D. iii) For the values a,b,c such that $\vec{F}$ is conservative on D find the potentials of $\vec{F}$.

EXERCISE 3.4.12. Let $a,b,\alpha,\beta\neq 0$ and $\vec{F}\in C^1(D)$ be the vector field

$$\vec{F}(x,y) := \left(\frac{ax}{(x^2+y^2)^\alpha}, \frac{by}{(x^2+y^2)^\beta}\right),\ \ (x,y)\in\mathbb{R}^2\backslash\{0_2\} =: D.$$

i) Find $a,b,\alpha,\beta\in\mathbb{R}\backslash\{0\}$ such that $\vec{F}$ be irrotational on D. ii) For the values found in i) compute $\int_\gamma \vec{F}$ where γ is the poligonal connecting $(2,0)$, $(0,1)$ and $(-2,0)$. iii) Find the values a,b,α,β such that $\vec{F}$ be conservative on D and compute the eventual potentials.

EXERCISE 3.4.13. Consider the vector field

$$\vec{F}(x,y) := \left(\frac{x}{\sqrt{x+y}}, \frac{ax+b}{\sqrt{x+y}}\right), \quad (x,y) \in D := \left\{(x,y) \in \mathbb{R}^2 \ : \ x+y > 0\right\}.$$

i) Find values $a, b \in \mathbb{R}$ such that $\vec{F}$ be irrotational. For such values may you say, without computing the potential, if $\vec{F}$ is also conservative? ii) For values $a, b \in \mathbb{R}$ such that $\vec{F}$ be conservative, find all its potentials.

EXERCISE 3.4.14. Let

$$\vec{F}(x,y,z) := \left(\frac{1}{x} + \frac{y^\alpha}{1+x^2y^2}, \frac{1}{y} + \frac{x}{1+x^2y^2}, \frac{1}{z}\right), \ (x,y,z) \in]0,+\infty[^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values α found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

EXERCISE 3.4.15. Let $\alpha \in \mathbb{R}$ and consider the vector field

$$\vec{F}(x,y,z) := \left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{x+\alpha z}{\sqrt{1+x^2+y^2}} + e^{y+z}, e^{y+z}\right), \ (x,y,z) \in \mathbb{R}^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values α found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

EXERCISE 3.4.16. Consider the vector field $\vec{F}(x,y) := \left(\frac{x}{x^2+y^2}, u(x,y)\right)$ on $D = \mathbb{R}^2\backslash\{0_2\}$, where $u \in \mathscr{C}^1(D)$. Find all the possible u in order that $\vec{F}$ be conservative.

EXERCISE 3.4.17. Find all the possible functions $u = u(x,y)$ belonging to $\mathscr{C}^1(\mathbb{R}^2)$ such that the vector field $\vec{F}(x,y,z) := (2xz, yz, u(x,y))$ be conservative in $D = \mathbb{R}^3$.

CHAPTER 4

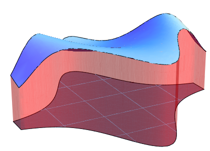
Integration

In the first course of Mathematical Analysis, the concept of *integral* for a function depending on one real variable has been introduced. Integration is of paramount relevance in Analysis and applications since it gives a method to solve geometrical problems (calculus of areas of figures), Probability, Physics, Engineering etc. We recall that if $f = f(x) : [a, b] \longrightarrow [0, +\infty[$,

$$\int_{[a,b]} f(x)\, dx = \text{Area}\,(\text{Trap}(f))\,, \quad \text{where } \text{Trap}(f) := \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b],\ 0 \leqslant y \leqslant f(x)\},$$

where AreaTrap(f) is defined through a complex *exhaustion method*. This Chapter extends this operation to the case of functions f of several variables. For instance, if $f = f(x, y) : E \subset \mathbb{R}^2 \longrightarrow [0, +\infty[$,

$$\int_E f(x, y)\, dxdy = \text{Volume}\,(\text{Trap}(f))\,, \text{ where } \text{Trap}(f) := \{(x, y, z) \in \mathbb{R}^3 \ : \ (x, y) \in E,\ 0 \leqslant z \leqslant f(x, y)\}.$$



More in general, given $f = f(\vec{x}) : E \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, we aim to define the integral

$$\int_E f(\vec{x})\, d\vec{x} = \text{measure}\left(\left\{(\vec{x}, y) \in \mathbb{R}^{m+1} \ : \ x \in E,\ 0 \leqslant y \leqslant f(\vec{x})\right\}\right).$$

Of course, how to compute this measure is the main problem of the construction. In the case of functions of one real variable we start defining

$$\underline{S}(\pi) := \textstyle\sum_{k=1}^n m_k(x_{k+1} - x_k), \quad \text{where } m_k := \inf_{x\in[x_k,x_{k+1}]} f(x),$$

$$\overline{S}(\pi) := \textstyle\sum_{k=1}^n M_k(x_{k+1} - x_k), \quad \text{where } M_k := \sup_{x\in[x_k,x_{k+1}]} f(x).$$

where $\pi = \{x_0 = a < x_1 < \ldots < x_n = b\}$ is a subdivision of $[a, b]$. $\underline{S}(\pi)$ and $\overline{S}(\pi)$ are, respectively, called inferior (superior) sum on subdivision π. These quantities represent approximations by defect (excess) of the area of Trap(f). Then, we define

$$\underline{A}(f) := \sup_\pi \underline{S}(\pi), \quad \text{(inferior area)},$$

$$\overline{A}(f) := \inf_\pi \overline{S}(\pi), \quad \text{(superior area)},$$

which are, respectively, the best approximations by defect (excess) of the area of $\mathrm{Trap}(f)$. Finally, we say that this area $A(f)$ is well defined when

$$\underline{A}(f) = \overline{A}(f) < +\infty.$$

This same procedure can be repeated for functions $f = f(\vec{x})$ with $\vec{x} \in \mathbb{R}^d$. We will sketch this in the first section. Once the area $A(f)$ of a trapezoid is defined, the definition of integral follows by an algebraic argument. As in the one dimensional case, the main issue is on methods of computing integrals. In the case of multiple integrals, two main tools are

- the *reduction formula*, that allows to transform the calculus of an integral for a function of m variables into the calculus of m integrals each on a single variable (this basically reduces calculus to the well known one dimensional integral)
- the *change of variables formula*, a well known technique with integrals that allows to simplify calculations under special coordinate systems.

Along this Chapter we will provide precise definitions and statements but we will omit all proofs. These are too technical and beyond our scope here. Nonetheless, we will provide informal justifications to the several results. Actually, these are the main ides behind the true proofs, without the technical complications to make them completely rigorous arguments. Yet, they are interesting to get some insight into this topic.

4.1. Measure of a trapezoid

In this section we define the operation of integral for a positive function. To prepare the ground, we introduce some useful definition. We call **multi-interval** of $\mathbb{R}^d$ any set

$$I = [a_1, b_1] \times \cdots \times [a_m, b_m].$$

The **measure** of a multi-interval is, by definition, the number

$$|I| := (b_1 - a_1) \cdots (b_m - a_m).$$

Notice that

- in dimension $m = 1$, a multi-interval is just an *interval* $[a, b]$, its measure is its *length* $|I| = b - a$;
- in dimension $m = 2$, a multi-inerval is a *rectangle* $I = [a_1, b_1] \times [a_2, b_2]$, its measure is its *area* $|I| = (b_1 - a_1)(b_2 - a_2)$;
- in dimension $m = 3$, a multi-interval is a *parallelepiped* $I = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, its measure is its *volume* $|I| = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$.

Let now $f = f(\vec{x}) : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, I a multi-interval.

Definition 4.1.1

A family $\pi := (I_j)_{j=1,\dots,n}$ is called partition of I if

i) $I = \bigcup_{k=1}^{n} I_k$;
ii) $|I_k \cap I_j| = 0$ for $j \neq k$.

We denote by $\Pi(I)$ the class made of all the partitions of I.

Notice that if I and J are multi-intervals, easily $I \cap J$ is a multi-interval thus $|I \cap J|$ is well defined.

Definition 4.1.2

Let $f : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, I interval. Given a partition $\pi \in \Pi(I)$ we set

$$\underline{S}(\pi) := \sum_{k=1}^n m_k |I_k|, \quad \text{where } m_k := \inf_{\vec{x} \in I_k} f(\vec{x}),$$

$$\overline{S}(\pi) := \sum_{k=1}^n M_k |I_k|, \quad \text{where } M_k := \sup_{\vec{x} \in I_k} f(\vec{x}).$$

$\underline{S}(\pi)$ and $\overline{S}(\pi)$ are called, respectively, **inferior sum** (**superior sum**) of the partition π.

As for one dimensional integral, $\underline{S}(\pi)$ and $\overline{S}(\pi)$ are, respectively, an approximation by defect (excess) of the measure of $\text{Trap}(f)$. We now introduce the best approximations by defect and excess:

Definition 4.1.3

$$\underline{A}(f) := \sup_{\pi \in \Pi(I)} \underline{S}(\pi), \quad (\textbf{inner measure of Trap}(f)),$$

$$\overline{A}(f) := \inf_{\pi \in \Pi(I)} \overline{S}(\pi), \quad (\textbf{outer measure of Trap}(f))$$

Easily, $\underline{A}(f) \leqslant \overline{A}(f)$. When they coincide, we say that

Definition 4.1.4

Let $f : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$. If $\underline{A}(f) = \overline{A}(f)$ we pose

$$\int_I f := \underline{A}(f) = \overline{A}(f) \in [0, +\infty]$$

Similarly to one dimensional definition, it may well happen that $\int_I f$ might not be defined:

Example 4.1.5 (Dirichlet function). *Let*

$$f = f(x, y) : [0, 1]^2 \longrightarrow [0, +\infty[, \quad f(x, y) := \begin{cases} 0, & (x, y) \in \mathbb{Q} \times \mathbb{Q}, \\ \\ 1, & (x, y) \notin \mathbb{Q} \times \mathbb{Q}. \end{cases}$$

Then, $\underline{A}(f) = 0 < 1 = \overline{A}(f)$.

Sol. — The argument is similar to the one dimensional case. If $\pi = (I_k)$ is a partition of $I = [0, 1]^2$, then, because of the density of rational numbers and irrational numbers in $\mathbb{R}$, certainly in each I_k there are points of $\mathbb{Q} \times \mathbb{Q}$ as well as points of $(\mathbb{R} \backslash \mathbb{Q}) \times (\mathbb{R} \backslash \mathbb{Q})$. Then $m_k = \inf_{I_k} f(x, y) = 0$ while $M_k = \sup_{I_k} f(x, y) = 1$. Therefore

$$\underline{S}(\pi) = 0, \quad \overline{S}(\pi) = \sum_k |I_k| = |I| = 1.$$

As a consequence, $\underline{A}(f) = 0$, while $\overline{A}(f) = 1$. □

Multi-intervals are not "natural" sets for $\mathbb{R}^m$ as intervals are in $\mathbb{R}$. A function $f = f(\vec{x}) : D \subset \mathbb{R}^m \longrightarrow [0, +\infty[$ can well be defined on a domain D which is not a multi-interval. We can easily extend previous definition to this case when D is bounded. In this case, there exists a multi-interval $I \supset D$. Defining

$$f1_D(x) := \begin{cases} f(x), & x \in D, \\ 0, & x \in I \backslash D, \end{cases}$$

we have

Definition 4.1.6

Let $f : D \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, D bounded. We pose

$$\int_D f := \int_I f 1_D.$$

It is possible to prove that previous definition does not depend of I. As the intuition suggests, since $f \geqslant 0$, if $D_1 \subset D_2$ then

$$\int_{D_1} f \leqslant \int_{D_2} f.$$

This leads to the idea to define $\int_D f$ when $D \subset \mathbb{R}^m$ is generic (also unbounded). Call

$$C_N := [-N, N]^m = [-N, N] \times \cdots \times [-N, N],$$

the hyper-cube centred at $\vec{0}$ with sides of length $2N$. Let

$$D_N := D \cap C_N.$$

Since $D_N \subset D_{N+1}$ we have

$$\int_{D_N} f \leqslant \int_{D_{N+1}} f.$$

This justifies the

Definition 4.1.7

Let $f = f(\vec{x}) : D \subset \mathbb{R}^m \longrightarrow [0, +\infty[$. We pose

$$\int_D f := \lim_{N \to +\infty} \int_{D_N} f.$$

This way, $\int_D f$ is now defined for $\geqslant 0$. In particular, taking $f \equiv 1$ on D we have the

Definition 4.1.8

Let $D \subset \mathbb{R}^m$. We call (m dimensional) **measure of** D the number

$$\lambda_m(D) := \int_D 1, \tag{4.1.1}$$

provided this last is well defined (in this case we say that D is **measurable**).

The following example, which is again the Dirichlet function under other form, tells that not every set is measurable:

Example 4.1.9. *Let* $D := \{(x, y) \in [0, 1]^2 \ : \ (x, y) \notin \mathbb{Q} \times \mathbb{Q}\}$. *Then* $\lambda_2(D)$ *is not defined.*

Sol. — Just notice that 1_D is the Dirichlet function, thus $\int_D 1 = \int_{[0,1]^2} 1_D$ which is not defined as shown in Example 4.1.5. □

Class of measurable sets is large enough to contain sets used in most of the applications:

Proposition 4.1.10

Open and closed sets are measurable.

We will see that, through techniques of calculus for integrals, calculus of measures is feasible in many applied cases.

4.2. Integral

So far, we defined the integral of a positive function. Let's now consider $f = f(\vec{x}) : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ and define

$$f_+(\vec{x}) := \begin{cases} f(\vec{x}), & \text{if } f(\vec{x}) \geqslant 0, \\ 0, & \text{if } f(\vec{x}) < 0, \end{cases} \qquad f_-(\vec{x}) := \begin{cases} -f(\vec{x}), & \text{if } f(\vec{x}) \leqslant 0, \\ 0, & \text{if } f(\vec{x}) > 0. \end{cases}$$

Functions $f_\pm$ are called, respectively, **positive part** and **negative part** of f. Both $f_\pm$ are positive and

$$f = f_+ - f_-, \quad |f| = f_+ + f_-.$$

We want to introduce the concept of *area with sign* or *integral of f*:

Definition 4.2.1

We say that f is **integrable** on D if $\int_D f_\pm < +\infty$. We pose

$$\int_D f := \int_D f_+ - \int_D f_-,$$

We write $f \in \mathscr{R}(D)$.

Notice that, since

$$\int_D f_+ + \int_D f_- = \int_D (f_+ + f_-) = \int_D |f|,$$

we have

$$\int_D f_\pm < +\infty, \iff \int_D |f| < +\infty.$$

In general, it is practically impossible to check if a given function is integrable by using the definition. This also happens with the one dimensional Riemann integral. In certain common cases, integrability can be easily drawn from good properties of function and integration domain:

Theorem 4.2.2

Let $f \in \mathscr{C}(D)$, D compact set. Then f is integrable on D.

In general, if D is not compact, in particular if D is closed and unbounded, continuity is not sufficient to ensure integrability (trivially, take $f \equiv 1$ on $D = \mathbb{R}^2$, then $\int_{\mathbb{R}^2} 1 = +\infty$). We have the following test

Proposition 4.2.3: absolute integrability

Let $f \in \mathscr{C}(D)$, D closed. Then, if

$$\int_D |f| < +\infty \tag{4.2.1}$$

f is integrable on D. When (4.2.1) holds we say that f is **absolutely integrable**.

For continuous function it is possible to prove that, if f is integrable on D, then for every $\varepsilon > 0$ there exists

- a family of multi-intervals $(I_k)_{k\in\mathbb{N}}$ such that $D \subset \bigcup_k I_k$;
- points $\vec{x}_k \in I_k \cap D$;

such that

$$\left| \int_D f - \sum_k f(\vec{x}_k)|I_k| \right| \leqslant \varepsilon. \tag{4.2.2}$$

This justifies the idea that, for good functions on good domains,

$$\int_D f \approx \sum_k f(\vec{x}_k) dx_1 \cdots dx_m.$$

Properties of the integral are very similar to those of one dimensional integral:

Proposition 4.2.4

The following properties hold true:

i) (linearity) if $f, g \in \mathscr{R}(D)$ then $\alpha f + \beta g \in \mathscr{R}(D)$ for any $\alpha, \beta \in \mathbb{R}$ and $\int_D (\alpha f + \beta g) = \alpha \int_D f + \beta \int_D g$;

ii) (isotonicity) if $f \leqslant g$ on D with $f, g \in \mathscr{R}(D)$ then $\int_D f \leqslant \int_D g$;

iii) (triangular inequality) if $f \in \mathscr{R}(D)$ then $\left|\int_D f\right| \leqslant \int_D |f|$;

iv) (decomposition) if $f \in \mathscr{R}(D_1), \mathscr{R}(D_2)$ with $D_1 \cap D_2 = \varnothing$, then $f \in \mathscr{R}(D_1 \cup D_2)$ and $\int_{D_1 \cup D_2} f = \int_{D_1} f + \int_{D_2} f$.

With this, the definition of integral is completed. Of course, as for one dimensional integral, we need efficient tools to check integrability and to compute an integral. The two most important tools are *reduction formula* and *change of variables formula*, which we illustrate in next sections.

4.3. Reduction formula

In this Section we introduce the technique based on the *reduction formula* that allows to reduce the calculus of a multiple variables integral to iterated one variable integrals. For pedagogical reasons we present first the case of double integrals, then we will extend to the general case.

4.3.1. Double Integrals. To understand the idea, let's consider the problem of computing

$$\int_D f(x, y)\, dxdy.$$

Assuming f continuous and integrable,

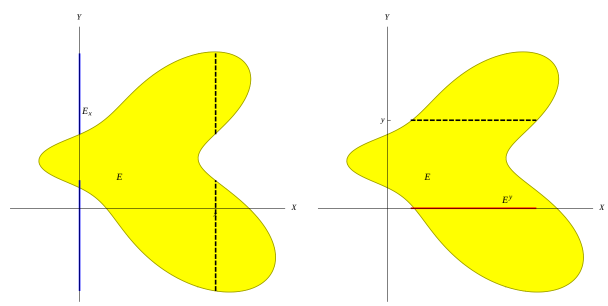
$$\int_D f(x,y)\, dxdy \approx \sum_{(x,y)\in D)} f(x,y) dxdy.$$

Informally, associative and commutative properties lead to

$$\sum_{(x,y)\in D)} f(x,y) dxdy = \sum_{x\in\mathbb{R}} \left(\sum_{y\,:\,(x,y)\in D} f(x,y)\, dy \right) dx.$$

Now,

$$\sum_{y\,:\,(x,y)\in D} f(x,y)\, dy \approx \int_{D_x} f(x,y)\, dy, \quad \text{where } D_x := \{y \;:\; (x,y)\in D\}.$$



We call D_x the $x-$section of D. Notice that D_x is the set of ordinates of points of D with abscissas $= x$. Thus, denoting by

$$F(x) := \int_{D_x} f(x,y)\, dy,$$

(this is a function of x, y is "integrated" and it does not appear out of the integral), we would have

$$\int_D f(x,y)\, dxdy \approx \sum_{x\in\mathbb{R}} F(x)\, dx \approx \int_{\mathbb{R}} F(x)\, dx \equiv \int_{\mathbb{R}} \left(\int_{D_x} f(x,y)\, dy \right)\, dy.$$

Similarly, flipping the role of x and y we have a similar formula with exchanged order of the integrations. Of course, this is not a proof, but the conclusion is a true fact:

Proposition 4.3.1: reduction formula

Let $f \in \mathscr{C}(D)$ be absolutely integrable on D. Then

(4.3.1) $$\int_D f(x,y)\, dxdy = \int_{\mathbb{R}} \left(\int_{D_x} f(x,y)\, dy \right)\, dx = \int_{\mathbb{R}} \left(\int_{D^y} f(x,y)\, dx \right)\, dy.$$

where

- $D_x := \{y \in \mathbb{R} \;:\; (x,y)\in D\}$;
- $D^y := \{x \in \mathbb{R} \;:\; (x,y)\in D\}$.

Notice that D_x (D^y) may be empty for certain values of x (y). For such x (y), clearly $\int_{D_x} f = 0$ ($\int_{D^y} f = 0$). Therefore, formula (4.3.1) and be actually written as

$$\int_D f(x,y)\,dxdy = \int_{x\,:\,D_x\neq\emptyset}\left(\int_{D_x} f(x,y)\,dy\right)\,dx = \int_{y\,:\,D^y\neq\emptyset}\left(\int_{D^y} f(x,y)\,dx\right)\,dy.$$

However, for future use we prefer to keep a lighter notation as in (4.3.1).

Reduction formula (RF) says that we can reduce the calculation of a "double integral" $\int_D f(x,y)\,dxdy$ to two iterated one variable integrals:

- first, one computes integral $\int_{D_x} f(x,y)\,dy$: the output is a function $F(x)$ of x;
- second, one computes integrale $\int_{\mathbb{R}} F(x)\,dx$.

Example 4.3.2. *Compute*

$$\int_D \cos(x+y)\,dxdy, \;\; \textit{where } D := \left\{(x,y)\in\mathbb{R}^2 \;:\; 0\leqslant y\leqslant x\leqslant\pi\right\}.$$

Sol. — First notice that $f(x,y)=\cos(x+y)\in\mathscr{C}(\mathbb{R}^2)$, thus $|f|\in\mathscr{C}(D)$, D is clearly closed and bounded hence compact. Therefore, f is absolutely integrable on D. To compute the integral we apply the RF. In this case, it is indifferent which one of the two forms, thus we write

$$\int_D \cos(x+y)\,dxdy = \int_{x\in\mathbb{R}}\left(\int_{y\in D_x}\cos(x+y)\,dy\right)\,dx.$$

Notice that

$$D_x = \{y \;:\; (x,y)\in D\} = \begin{cases} [0,x], & x\in[0,\pi]\,, \\ \emptyset, & x\notin[0,\pi]\,. \end{cases}$$

Thus

$$\int_D \cos(x+y)\,dxdy = \int_0^{\pi}\left(\int_0^x \cos(x+y)\,dy\right)\,dx.$$

Now,

$$\int_0^x \cos(x+y)\,dy = [\sin(x+y)]_{y=0}^{y=x} = \sin(2x)-\sin x,$$

thus

$$\int_D \cos(x+y)\,dxdy = \int_0^{\pi}(\sin(2x)-\sin x)\,dx = \left[-\frac{\cos(2x)}{2}+\cos x\right]_{x=0}^{x=\pi} = \left(-\frac{1}{2}-1\right)-\left(-\frac{1}{2}+1\right) = -2. \qquad \square$$

RF (4.3.1) requires f absolutely integrable. This means to check that

$$\int_D |f(x,y)|\,dxdy < +\infty.$$

To check this, in principle one should compute a double integral. Applying the reduction formula to $|f|$,

$$\int_D |f(x,y)|\,dxdy = \int_{\mathbb{R}}\left(\int_{D_x}|f(x,y)|\,dy\right)\,dx = \int_{\mathbb{R}}\left(\int_{D^y}|f(x,y)|\,dx\right)\,dy,$$

so, in particular if f is integrable

$$\int_{\mathbb{R}}\left(\int_{E_x}|f(x,y)|\,dy\right)\,dx,\; \int_{\mathbb{R}}\left(\int_{E^y}|f(x,y)|\,dx\right)\,dy \;<+\infty.$$

It turns out that also the vice versa holds true:

Proposition 4.3.3

Let $f \in \mathscr{C}(D)$ on D closed or open. If one of the following iterated integrals

$$\int_{\mathbb{R}} \left(\int_{D_x} |f(x,y)| \, dy \right) dx, \quad \int_{\mathbb{R}} \left(\int_{D^y} |f(x,y)| \, dx \right) dy \tag{4.3.2}$$

is finite, then f is absolutely integrable on D and reduction formula (4.3.1) holds.

Combining the previous Propositions we have an algorithm to check integrability and compute double integrals: *to compute $\int_D f(x,y)\, dxdy$ we have*

- *first, to check that at least one of the iterated integrals (4.3.2) is finite;*
- *second, apply RF (4.3.1) to compute the integral.*

Notice that, if $f \geqslant 0$ the first step, if the outcome is finite, automatically gives the value of the second step.

Example 4.3.4. *Discuss if $f(x,y) := x^3 e^{-yx^2}$ is integrable on $D = [0,+\infty[\times[1,2]$ and, in this case, compute its integral.*

Sol. — Clearly $f \in \mathscr{C}(D)$ where $D = [0,+\infty[\times[1,2]$ is closed. Trivially,

$$D_x = \begin{cases} \emptyset, & x < 0, \\ [1,2], & x \geqslant 0. \end{cases}$$

Moreover, $f \geqslant 0$ on D, thus $|f| = f$ and

$$\begin{aligned} \int_{\mathbb{R}} \int_{D_x} |f| \, dy \, dx &= \int_0^{+\infty} \left(\int_1^2 x^3 e^{-yx^2} \, dy \right) dx = \int_0^{+\infty} x \left[-e^{-yx^2} \right]_{y=1}^{y=2} dx \\ &= \int_0^{+\infty} xe^{-x^2} - xe^{-2x^2} \, dx = \left[\frac{-e^{-x^2}}{2} \right]_{x=0}^{x=+\infty} - \left[\frac{-e^{-2x^2}}{4} \right]_{x=0}^{x=+\infty} = \frac{1}{4}. \end{aligned}$$

We deduce f is integrable and being $f \geqslant 0$, the previous calculation provides also $\int_{[0,+\infty[\times[1,2]} f = \frac{1}{4}$. □

Example 4.3.5. *Discuss if $f(x,y) := e^{-x}$ is integrable on $D := \{(x,y) \in \mathbb{R}^2 \,:\, x \geqslant 0,\ 0 \leqslant y \leqslant x^2\}$. In such case compute the integral of f on D.*

Sol. — Clearly $f \in \mathscr{C}(D)$ and D is closed (defined by large inequalities on continuous functions). Applying (4.3.2), notice that

$$D_x = \begin{cases} \emptyset, & x < 0, \\ [0,x^2], & x \geqslant 0. \end{cases}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}\int_{D_x} |f|\ dy\ dx &= \int_0^{+\infty}\left(\int_0^{x^2} e^{-x}\ dy\right)\ dx = \int_0^{+\infty} x^2 e^{-x}\ dx = \int_0^{+\infty} x^2(-e^{-x})'\ dx \\ &= \left[-x^2e^{-x}\right]_{x=0}^{x=+\infty} + \int_0^{+\infty} 2xe^{-x}\ dx = 2\int_0^{+\infty} x(-e^{-x})'\ dx \\ &= 2\left[[-xe^{-x}]_{x=0}^{x=+\infty} + \int_0^{+\infty} e^{-x}\ dx\right] = 2\,[-e^{-x}]_{x=0}^{x=+\infty} = 2.\end{aligned}$$

This says that f is integrable on D and, at same time, $\int_D f = 2$. □

A particular case of RF (4.3.1) is obtained by taking $f \equiv 1$. Recalling that $\int_D 1 = \lambda_2(D)$ we obtain

$$\lambda_2(D) = \int_{\mathbb{R}}\left(\int_{D_x} 1\ dy\right)\ dx = \int_{\mathbb{R}} \lambda_1(D_x)\ dx = \int_{\mathbb{R}} \lambda_1(D^y)\ dy. \tag{4.3.3}$$

Formula (4.3.3) is called *slicing formula*.

Example 4.3.6. *Compute the area of a disk of radius r.*

Sol. — Let

$$D = \left\{(x, y) \in \mathbb{R}^2\ :\ x^2 + y^2 \leqslant r^2\right\}.$$

This set D is closed, hence measurable. According to the slicing formula,

$$\lambda_2(D) = \int_{\mathbb{R}} \lambda_1(D_x)\ dx.$$

Let's determine an $x-$section. We have

$$y \in D_x, \iff (x, y) \in D, \iff x^2+y^2 \leqslant r^2, \iff y^2 \leqslant r^2-x^2, \iff \begin{cases} \emptyset, & |x| > r, \\ \left[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}\right], & |x| \leqslant r. \end{cases}$$

Therefore

$$\lambda_2(D) = \int_{\mathbb{R}} \lambda_1(D_x)\ dx = \int_{|x|\leqslant r} \lambda_1\left(\left[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}\right]\right)\ dx = \int_{|x|\leqslant r} 2\sqrt{r^2-x^2}\ dx.$$

Setting $x = r\sin\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have

$$\lambda_2(D) = 2r\int_{-\pi/2}^{\pi/2} \sqrt{1-(\sin\theta)^2}\ r\cos\theta\ d\theta = 2r^2\int_{-\pi/2}^{\pi/2} (\cos\theta)^2\ d\theta.$$

Now $\int(\cos\theta)^2 = \int\cos\theta(\sin\theta)' = \cos\theta\sin\theta + \int(\sin\theta)^2 = \frac{1}{2}\sin(2\theta) + \theta - \int(\cos\theta)^2$ hence

$$\lambda_2(D) = 4r^2\left[\frac{1}{4}\sin(2\theta) + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \pi r^2. \quad \square$$

Warning! It might well happen that both iterated integrals of RF make sense but they are different! Of course, in this case, f cannot be absolutely integrable (otherwise they should coincide).

Example 4.3.7. *Let*

$$f(x,y) = \frac{x-y}{(x+y)^3},\ (x,y)\in D := [0,1]^2.$$

Then $\int_{\mathbb{R}}\left(\int_{D_x} f\ dy\right)\ dx \neq \int_{\mathbb{R}}\left(\int_{D^y} f\ dx\right)\ dy.$

Sol. — Notice first that

$$D_x = \{y\in\mathbb{R}\ :\ (x,y)\in[0,1]^2\} = \begin{cases} \emptyset, & x\notin[0,1],\\ \\ [0,1] & x\in[0,1]\end{cases}$$

and similarly for D^y. Therefore

$$\int_{D^y} f(x,y)\ dx = \begin{cases} 0, & y\notin[0,1],\\ \\ \displaystyle\int_0^1 \frac{x-y}{(x+y)^3}\ dx = \int_0^1 \frac{1}{(x+y)^2}\ dx - 2y\int_0^1 \frac{1}{(x+y)^3}\ dx. & y\in[0,1].\end{cases}$$

Apart for $y=0$, both integrals are finite equal to

$$\left[\frac{(x+y)^{-1}}{-1}\right]_{x=0}^{x=1} - 2y\left[\frac{(x+y)^{-2}}{-2}\right]_{x=0}^{x=1} = \frac{1}{y} - \frac{1}{y+1} + y\left(\frac{1}{(y+1)^2} - \frac{1}{y^2}\right) = -\frac{1}{(y+1)^2}.$$

Hence

$$\int_{\mathbb{R}}\left(\int_{D^y} f(x,y)\ dx\right)\ dy = \int_0^1\left(-\frac{1}{(y+1)^2}\right)\ dy = \left[(y+1)^{-1}\right]_{y=0}^{y=1} = \frac{1}{2} - 1 = -\frac{1}{2}.$$

Exchanging x with y we obtain the same result except for the sign: $\int_{\mathbb{R}}\left(\int_{D_x} f(x,y)dy\right)\ dx = \frac{1}{2}$. □

4.3.2. General Multiple Integrals. The previous mechanism can be extended to functions f of m variables. Let $f = f(x_1,\dots,x_m)$, and imagine we group $(x_1,\dots,x_m)$ into two blocks, one of k variables and the remaining of $m-k$ variables, that is

$$(x_1,\dots,x_m) = (\underbrace{x_1,\dots,x_k}_{\vec{x}},\underbrace{x_{k+1},\dots,x_m}_{\vec{y}}) \equiv (\vec{x},\vec{y}),\ \vec{x}\in\mathbb{R}^k,\ \vec{y}\in\mathbb{R}^{m-k}.$$

With this notation we may write

$$f(x_1,\dots,x_m) = f(\vec{x},\vec{y}).$$

We have the

Theorem 4.3.8

Let $f \in \mathscr{C}(D)$ be absolutely integrable on D closed or open in $\mathbb{R}^m$. Then

$$\int_D f = \int_{\mathbb{R}^k} \left(\int_{D_{\vec{x}}} f(\vec{x}, \vec{y})\, d\vec{y} \right) d\vec{x} = \int_{\mathbb{R}^{m-k}} \left(\int_{D^{\vec{y}}} f(\vec{x}, \vec{y})\, d\vec{x} \right) d\vec{y}. \tag{4.3.4}$$

Moreover, if one of integrals

$$\int_{\mathbb{R}^k} \left(\int_{D_{\vec{x}}} |f(\vec{x}, \vec{y})|\, d\vec{y} \right) d\vec{x}, \quad \int_{\mathbb{R}^{m-k}} \left(\int_{D^{\vec{y}}} |f(\vec{x}, \vec{y})|\, d\vec{x} \right) d\vec{y},$$

is finite, then f is absolutely integrable on D (and the reduction formula (4.3.4) holds). In particular, by taking $f = 1$ we have the **slicing formula**

$$\lambda_m(D) = \int_{\mathbb{R}^k} \lambda_{m-k}(D_{\vec{x}})\, d\vec{x} = \int_{\mathbb{R}^{m-k}} \lambda_m(D^{\vec{y}})\, d\vec{y}. \tag{4.3.5}$$

Remark 4.3.9. Consider a function of three variables $f = f(x, y, z) \in \mathscr{C}(D)$, $D \subset \mathbb{R}^3$ open/closed. In this common case, the three variables may be grouped is six different ways, this leading to six different possible applications of reduction formula:

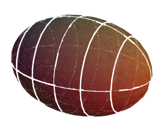
$$x \text{ and } (y,z), \quad \int_D f = \int_{\mathbb{R}} \left(\int_{(y,z)\in D_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x \in D_{(y,z)}} f\, dx \right) dydz,$$

$$y \text{ and } (x,z), \quad \int_D f = \int_{\mathbb{R}} \left(\int_{(x,z)\in D_y} f\, dxdz \right) dy = \int_{\mathbb{R}^2} \left(\int_{y \in D_{(x,z)}} f\, dy \right) dxdz,$$

$$z \text{ and } (x,y), \quad \int_D f = \int_{\mathbb{R}} \left(\int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x \in D_{(y,z)}} f\, dx \right) dydz,$$

Which choice is the best one depends by the complexity of calculus. □

Example 4.3.10. *Compute the volume of a rugby ball* $D := \left\{ (x, y, z) \in \mathbb{R}^3 \;:\; \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1 \right\}$, $(a, b > 0)$.



Sol. — Clearly D is closed and bounded in $\mathbb{R}^3$, hence measurable. Slicing D along the z-axis,

$$\lambda_3(D) = \int_{\mathbb{R}} \lambda_2(D_z)\, dz.$$

Now,

$$(x, y, z) \in D, \iff \frac{x^2+y^2}{a^2} \leqslant 1 - \frac{z^2}{b^2}, \iff \begin{cases} \emptyset, & |z| > b, \\ B\left(0_2, \sqrt{1 - \frac{z^2}{b^2}}\right], & |z| \leqslant b. \end{cases}$$

Thus

$$\lambda_3(E) = \int_{|z|\leqslant b} \lambda_2\left(B\left(0_2, a\sqrt{1-\frac{z^2}{b^2}}\right]\right)\, dz = \int_{|z|\leqslant b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\, dz = \int_{-b}^{b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\, dz$$

$$= \pi a^2\left([z]_{-b}^{b} - \left[\frac{z^3}{3b^2}\right]_{-b}^{b}\right) = \pi a^2\left(2b - \frac{2}{3}b\right) = \pi\frac{4}{3}a^2 b.$$

Taking $a = b = r$ we obtain the volume of a sphere of radius r, the well known $\frac{4}{3}\pi r^3$. $\square$

4.4. Change of variable

Change of variable is an important technique of calculus for integrals. We recall that if we have to compute

$$\int_a^b f(x)\, dx$$

for an $f \in \mathscr{C}([a,b])$ and supposing that, for convenience we wish to set $y = \phi(x)$, with $\phi \in \mathscr{C}^1$ a regular bijection such that $\phi^{-1} \in \mathscr{C}^1$ as well (thus $x = \phi^{-1}(y)$), then

$$\int_a^b f(x)\, dx = \begin{cases} \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))(\phi^{-1})'(y)\, dy, & \text{if } \phi \nearrow, \\ \\ \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))(\phi^{-1})'(y)\, dy, & \text{if } \phi \searrow. \end{cases}$$

Denoting with $\phi([a,b])$ the image of $[a,b]$ through ϕ, we have, in a unique formula

$$\int_a^b f(x)\, dx = \int_{\phi([a,b])} f(\phi^{-1}(y))|(\phi^{-1})'(y)|\, dy.$$

Suppose now we have to compute

$$\int_D f(\vec{x})\, d\vec{x},$$

for $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and we wish to introduce a new variable $\vec{y} = \Phi(\vec{x})$. We have the

Theorem 4.4.1

Let $f \in \mathscr{C}(D)$, D closed or open domain in $\mathbb{R}^d$. Suppose that $\Phi : D \longrightarrow F = \Phi(D)$ is a **diffeomorphism on** D that is

- Φ is a bijection: $\exists \Phi^{-1} : F \longrightarrow D$;
- Φ, Φ^{-1} are *regular*, that is $\Phi, \Phi^{-1} \in \mathscr{C}^1$ on their domains.

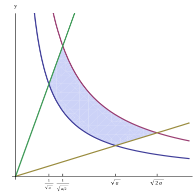
Then

$$\boxed{\int_D f(\vec{x})\, d\vec{x} \overset{\vec{y}=\Phi(\vec{x}),\ \vec{x}=\Phi^{-1}(\vec{y})}{=} \int_{\Phi(D)} f(\Phi^{-1}(\vec{y}))|\det(\Phi^{-1})'(\vec{y})|\, d\vec{y}.} \tag{4.4.1}$$

Example 4.4.2. *Compute*

$$\int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy, \quad 0 < a < 1.$$

Sol. — The domain is closed in $\mathbb{R}^2$ and $f \in \mathscr{C}$.



Notice that

$$\frac{y^4 \arctan(xy)}{(x^2+y^2)^2} = \left(\frac{y}{x}\right)^4 \frac{\arctan(xy)}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}.$$

It seems therefore natural to introduce the new variables

$$\xi = xy, \quad \eta = \frac{y}{x}, \quad (\xi, \eta) := \Phi(x, y),$$

where $\Phi :]0,+\infty[^2 \longrightarrow]0,+\infty[^2$, $\Phi(x,y) = (xy, \frac{y}{x})$ is clearly $\mathscr{C}^1$. We need Φ^{-1}. If $(\xi,\eta) \in]0,+\infty[^2$ then

$$\left\{ \begin{array}{l} \xi = xy, \\ \\ \eta = \frac{y}{x}, \end{array} \right. \iff \left\{ \begin{array}{l} \xi = \eta x^2, \\ \\ y = \eta x, \end{array} \right. \iff \left\{ \begin{array}{l} x = \sqrt{\frac{\xi}{\eta}}, \\ \\ y = \sqrt{\xi\eta}, \end{array} \right. \iff \Phi^{-1}(\xi,\eta) = \left(\sqrt{\frac{\xi}{\eta}}, \sqrt{\xi\eta}\right).$$

Therefore

$$I(a) := \int_{1 \leqslant xy \leqslant 2,\ 0 < ax \leqslant y \leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy = \int_{1 \leqslant \xi \leqslant 2,\ a \leqslant \eta \leqslant \frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi \, |\det(\Phi^{-1})'(\xi,\eta)|\, d\xi d\eta.$$

and because

$$|\det(\Phi^{-1})'(\xi,\eta)| = \frac{1}{|\det \Phi'(\Phi^{-1}(\xi,\eta))|},$$

with

$$\Phi'(x,y) = \begin{bmatrix} y & x \\ \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}, \implies \det \Phi'(x,y) = \frac{y}{x} + x\frac{y}{x^2} = 2\frac{y}{x} = 2\eta.$$

we have

$$I(a) = \int_{1 \leqslant \xi \leqslant 2,\ a \leqslant \eta \leqslant \frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi \frac{1}{2\eta}\, d\xi d\eta = \frac{1}{2}\left(\int_1^2 \arctan\xi\, d\xi\right)\left(\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta\right).$$

Now

$$\int_1^2 \arctan\xi\, d\xi = [\xi \arctan\xi]_1^2 - \int_1^2 \frac{\xi}{1+\xi^2}\, d\xi = 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}\log\frac{5}{2},$$

while

$$\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta = \int_a^{\frac{1}{a}} \frac{\eta}{1+\eta^2}\, d\eta - \int_a^{\frac{1}{a}} \frac{\eta}{(1+\eta^2)^2}\, d\eta = -\log a + \frac{1}{2}\frac{1-a^2}{1+a^2}. \qquad \square$$

4.4.1. Polar coordinates in $\mathbb{R}^2$. A very important change of variable in plane integration is

$$\begin{cases} x = \rho\cos\theta, \\ \\ y = \rho\sin\theta, \end{cases} \iff (x, y) = \Psi(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta).$$

Here we may notice that change of variable is defined in the form $(x, y) = \Psi(\rho, \theta)$. This means that, referring to notations of (4.4.1), present Ψ is just Φ^{-1}. Thus

$$\det(\Phi^{-1})' = \det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \\ \sin\theta & \rho\cos\theta \end{bmatrix} = \rho(\cos^2\theta + \sin^2\theta) = \rho,$$

and (4.4.1) becomes

$$\int_E f(x, y)\, dxdy = \int_{E_{pol}} f(\rho\cos\theta, \rho\sin\theta)\rho\, d\rho d\theta, \tag{4.4.2}$$

where E_{pol} is E in polar coordinates.

Example 4.4.3. *Compute*

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dxdy.$$

Sol. — We have

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dxdy &= \int_{\rho\geqslant 0,\, \theta\in[0,2\pi]} e^{-\rho}\rho\, d\rho d\theta = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\rho}\rho\, d\theta\right)\, d\rho = 2\pi\int_0^{+\infty}\rho e^{-\rho}\, d\rho \\ &= 2\pi\left([-\rho e^{-\rho}]_{\rho=0}^{\rho=+\infty} + \int_0^{+\infty} e^{-\rho}\, d\rho\right) = 2\pi. \quad \square \end{aligned}$$

Next one is a very smart calculation:

Example 4.4.4 (Gaussian integral). *We have*

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}.$$

More in general: if C is a $m\times m$ positive symmetric matrix,

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}C^{-1}x\cdot x}\, dx = \sqrt{(2\pi)^m\det C}. \tag{4.4.3}$$

Sol. — Let's start by the integral

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{x^2+y^2}{2}}\, dx\right)\, dy = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dx\right)\, dy = \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx\right)^2.$$

On the other hand, by (4.4.2)

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\frac{\rho^2}{2}}\rho\, d\theta\right)\, d\rho = 2\pi\int_0^{+\infty} e^{-\frac{\rho^2}{2}}\rho\, d\rho = 2\pi\left[e^{-\frac{\rho^2}{2}}\right]_{\rho=0}^{\rho=+\infty} = 2\pi,$$

and by this the conclusion follows.

To compute (4.4.3) notice first that, being C symmetric, it is diagonalizable: this means that there exists T invertible such that $T^{-1}CT = \mathrm{diag}(\sigma_1, \dots, \sigma_d)$. Furthermore, because C is symmetric, T is also orthogonal, that is $T^{-1} = T^t$ (transposed matrix). Therefore $C = TDT^{-1}$, hence

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}C^{-1}\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}(TDT^{1-})^{-1}\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}(TD^{-1}T^{1-})\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}T^{-1}\vec{x}\cdot T^{-1}\vec{x}}\, d\vec{x}.$$

Now, set $\vec{y} = T^{-1}\vec{x}$, in such a way that $\vec{x} = T\vec{y}$ and

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}T^t\vec{x}\cdot T^t\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\, |\det T|\, d\vec{y} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\, d\vec{y}.$$

Last $=$ is justified because, being T orthogonal, $TT^t = \mathbb{I}$, hence $1 = \det(TT^t) = \det T \det T^t = (\det T)^2$ by which $|\det T| = 1$. Moreover,

$$D^{1-}\vec{y}\cdot\vec{y} = \sum_j \frac{1}{\sigma_j} y_j^2,$$

therefore

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\, d\vec{y} = \int_{\mathbb{R}^m} \prod_{j=1}^m e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j = \prod_{j=1}^m \int_{\mathbb{R}} e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j \overset{x_j = \frac{y_j}{\sqrt{\sigma_j}}}{=} \prod_{j=1}^m \sqrt{\sigma_j} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{(2\pi)^m \sigma_1 \cdots \sigma_m}.$$

To conclude just notice that

$$\sigma_1 \cdots \sigma_m = \det D = \det(T^{-1}CT) = \det T^{-1} \det C \det T = \det C. \qquad \square$$

4.4.2. Spherical and cylindrical coordinates. The analogous of polar coordinates for functions of three variables are *spherical coordinates*:

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \qquad (\rho, \theta, \varphi) \in [0, +\infty[\times[0, 2\pi] \times [0, \pi]. \\ z = \rho\cos\varphi. \end{cases}$$

Also in this case the change of variable is defined in the form

$$(x, y, z) = \Psi(\rho, \theta, \varphi),$$

thus, referring to (4.4.1), $\Psi = \Phi^{-1}$. Hence,

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} = \rho^2\sin\varphi.$$

Therefore, (4.4.1) reads as

$$\int_E f(x, y, z)\, dxdydz = \int_{E_{spher}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho.$$

Here E_{spher} is E in spherical coordinates. This type of change of variable is often useful when f has some spherical symmetry, that is it depends on $x^2 + y^2 + z^2$.

EXAMPLE 4.4.5. *Using spherical coordinates, compute the volume of a sphere of radius r.*

SOL. — We have

$$\lambda_3\left(\{x^2+y^2+z^2 \leqslant r^2\}\right) = \int_{x^2+y^2+z^2\leqslant r^2} dxdydz = \int_{0\leqslant\rho\leqslant r,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} \rho^2 \sin\varphi\ d\rho d\theta d\varphi$$

$$= 2\pi\left(\int_0^\pi \sin\varphi\ d\varphi\right)\left(\int_0^r \rho^2\ d\rho\right) = \frac{4}{3}\pi r^3. \quad \square$$

When f has not a central symmetry but it is symmetric respect to some of the axes, a further variant of polar coordinates may be useful. Let first introduce this system of coordinates defined as

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \qquad (\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R}. \\ z = z. \end{cases}$$

Also in this case the change of variables is defined in the form

$$(x,y,z) = \Psi(\rho,\theta,z), \ \text{ where } \Psi = \Phi^{-1}.$$

Being,

$$\det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho,$$

according to (4.4.1) we have

$$\int_E f(x,y,z)\ dxdydz = \int_{E_{cil}} f(\rho\cos\theta,\rho\sin\theta,z)\rho\ d\rho d\theta dz.$$

This change of variables is particularly useful in the case of functions symmetric respect to the z axis (that is depending on x^2+y^2 that becomes ρ^2 in new coords).

EXAMPLE 4.4.6. *Compute the volume of the rugby ball* $E = \{(x,y,z) \in \mathbb{R}^3 \ :\ \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1\}$ *by adapting cylindrical coordinates.*

PROOF. Adapting the cylindrical coords $(x,y,z) = \Psi^{-1}(\rho,\theta,z) := (a\rho\cos\theta, a\rho\sin\theta, bz)$ we have

$$\det(\Psi^{-1})' = \det\begin{bmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ a\sin\theta & a\rho\cos\theta & 0 \\ 0 & 0 & b \end{bmatrix} = ba^2\rho,$$

therefore

$$\lambda_3(E) = \int_{\rho^2+\tilde{z}^2\leqslant 1,\ \rho\geqslant 0,\ \theta\in[0,2\pi],\ \tilde{z}\in\mathbb{R}} ba^2\rho\ d\rho d\theta dz = 2\pi a^2 b\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho\ d\rho dz.$$

To compute the last integral we may use polar coords for $(\rho,z) = (r\cos\alpha, r\sin\alpha)$. Then

$$\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho\ d\rho dz = \int_{-\frac{\pi}{2}\leqslant\alpha\leqslant\frac{\pi}{2},\ 0\leqslant r\leqslant 1} (r\cos\alpha)r\ drd\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha\ d\alpha\int_0^1 r^2\ dr = \frac{2}{3}.$$

Moral: $\lambda_3(E) = \frac{4\pi}{3}a^2 b.$ $\square$

4.5. Barycenter, center of mass, inertia moments

Through multiple integrals we can define several quantities relevant in Geometry and Physics. To fix ideas consider a set $D \subset \mathbb{R}^3$. We call **barycenter** of D the point $(\overline{x}, \overline{y}, \overline{z})$ defined as

$$\overline{x} = \frac{1}{\lambda_3(D)} \int_D x \, dx \, dy \, dz, \quad \overline{y} = \frac{1}{\lambda_3(D)} \int_D y \, dx \, dy \, dz, \quad \overline{z} = \frac{1}{\lambda_3(D)} \int_D z \, dx \, dy \, dz.$$

In other words, the barycenter is the point whose coords are the mean values of the coords of D. With special symmetries some of the coords of the barycenter may vanish. For instance, if D is symmetric with respect to the plane yz, that is $(x, y, z) \in D$ iff $(-x, y, z) \in D)$, then $\overline{x} = 0$. Indeed, if $\Phi(x, y, z) = (-x, y, z)$ we have $\Phi(D) = D$ and $\det \Phi' = 1$, therefore, by change of variables,

$$\int_D x \, dxdydz = \int_{\Phi(D)} x \, dxdydz = \int_D (-x) |\det \Phi'(x, y, z)| \, dxdydz = -\int_D x \, dxdydz$$

from which it follows that $\int_D x \, dxdydz = 0$.

If D represents a solid body with density of mass $\varrho = \varrho(x, y, z)$, the total mass is, by definition,

$$\mu(D) := \int_D \varrho(x, y, z) \, dxdydz.$$

In Physics it is important the **center of mass**: it is the point where the sum of all the forces acting on D could be applied to get the same effect. This point has coords (x_G, y_G, z_G)

$$x_G = \frac{1}{\mu(D)} \int_D x\varrho(x, y, z) \, dxdydz, \quad y_G = \frac{1}{\mu(D)} \int_D y\varrho(x, y, z) \, dxdydz, \quad z_G = \frac{1}{\mu(D)} \int_D z\varrho(x, y, z) \, dxdydz.$$

If the body is homogeneous (that is $\varrho \equiv \varrho_0 \in \mathbb{R}$) the center of mass coincide with the barycenter, as it is easy to verify.

Another important quantity for Physics is the **inertia moment with respect to some axis**. For instance, if the axis is the z one, this is defined by

$$I_z := \int_D (x^2 + y^2)\varrho(x, y, z) \, dxdydz.$$

Example 4.5.1. *Determine the barycenter of a spherical cap $E := \{(x, y, z) \ : \ x^2 + y^2 + z^2 \leqslant r^2, \ z \geqslant h\}$ with $0 \leqslant h < r$.*

Sol. — By symmetry, it is evident that $\overline{x} = \overline{y} = 0$. Let's compute

$$\overline{z} = \frac{1}{\lambda_3(D)} \int_D z \, dxdydz.$$

It seems convenient to slice D perpendicularly to the z−axis:

$$\begin{aligned} \lambda_3(D) \quad &= \int_h^r \left(\int_{x^2+y^2 \leqslant r^2 - z^2} dxdy \right) \, dh = \int_h^r \pi(r^2 - z^2) \, dz = \pi r^2 (r - h) - \pi \left[\frac{z^3}{3} \right]_{z=h}^{z=r} \\ &= \pi(r - h) \left(r^2 - \frac{1}{3}(r^2 + rh + h^2) \right). \end{aligned}$$

Similarly

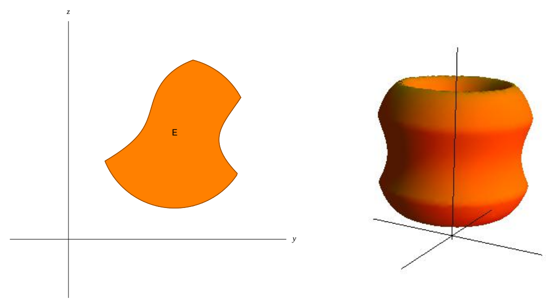
$$\begin{aligned}\int_D z\;dxdydz &= \int_h^r \left(\int_{x^2+y^2\leqslant r^2-z^2} z\;dxdy\right)\,dz = \int_h^r z\left(\int_{x^2+y^2\leqslant r^2-z^2} dxdy\right)\,dz = \int_h^r z\pi(r^2-z^2)\;dz\\ &= \pi r^2\left[\frac{z^2}{2}\right]_{z=h}^{z=r} - \pi\left[\frac{z^4}{4}\right]_{z=h}^{z=r} = \pi r^2\frac{r^2-h^2}{2} - \pi\frac{r^4-h^4}{4} = \pi\frac{r^2-h^2}{2}\left(r^2-\frac{r^2+h^2}{2}\right)\\ &= \pi\frac{(r^2-h^2)^2}{4}.\end{aligned}$$

By this we get $\overline{z}$. In the case $h=0$ (that is when D is the half-sphere) we have $\overline{z}=\frac{3}{8}r$. □

Let $D\subset\mathbb{R}^3$ by a domain obtained by a rotation around one of the axes of a plane set E. To fix ideas, let's assume that the rotation be around the $z-$axis of a domain E in the plane yz. This domain can be identified by $\{(0,y,z)\ :\ (y,z)\in E\}\subset\mathbb{R}^3$. Therefore, D can be represented as

$$D=\{(y\cos\theta,y\sin\theta,z)\ :\ (y,z)\in E,\ \theta\in[0,2\pi]\}=\Phi(E\times[0,2\pi]),$$

where Φ is nothing but the cylindrical coordinates map.



By the formula of change of variables

$$\lambda_3(D)=\int_{E\times[0,2\pi]}|\Phi'(y,\theta,z)|\;dy\;d\theta\;dz=\int_{E\times[0,2\pi]} y\;dy\;d\theta\;dz=2\pi\int_E y\;dydz,$$

that gives the **Pappo's Theorem**:

$$\lambda_3(D)=2\pi\lambda_2(E)\overline{y}. \tag{4.5.1}$$

Example 4.5.2. *Compute the volume of a thorus* $\mathbb{T}_{r,R}:=\left\{(x,y,z)\in\mathbb{R}^3\ :\ (\sqrt{x^2+y^2}-R)^2+z^2\leqslant r^2\right\}$ *(*$0<r<R$*).*

Sol. — According to Pappo's formula (4.5.1), we have

$$\lambda_3(\mathbb{T}_{r,R})=2\pi\lambda_2\left(\{(y-R)^2+z^2\leqslant r^2\right)\overline{y}=2\pi 1\pi r^2\overline{y}=4\pi^2r^2\overline{y}.$$

Here $\overline{y}$ it's the ordinate of the barycenter of the disk $E:=\{(y-R)^2+z^2\leqslant r^2\}$, so

$$\overline{y}=\frac{1}{\lambda_2(E)}\int_E y\;dydz=\frac{1}{\pi r^2}\int_{(y-R)^2+z^2\leqslant r^2} y\;dydz.$$

Changing to polar coord $y-R=\rho\cos\theta$, $z=\rho\sin\theta$, we have easily

$$\overline{y}=\frac{1}{\pi r^2}\int_0^{2\pi}\left(\int_0^r \rho(R+\rho\cos\theta)\;d\rho\right)\;d\theta=\frac{1}{\pi r^2}2\pi\frac{r^2}{2}R=R,$$

(as it is natural!). Hence $\lambda_3(\mathbb{T}_{r,R}) = 4\pi^2 r^2 R$. □

4.6. Green formula

Green formula is a remarkable application of multiple integrals to vector fields. Let $\vec{F} = (F_1, F_2) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a two dimensional vector field.

Theorem 4.6.1

Let $\vec{F} = (F_1, F_2) : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a vector field. Let $\gamma = \partial D$ where $D \subset \Omega$ is open. Then, if γ is counterclockwise oriented,

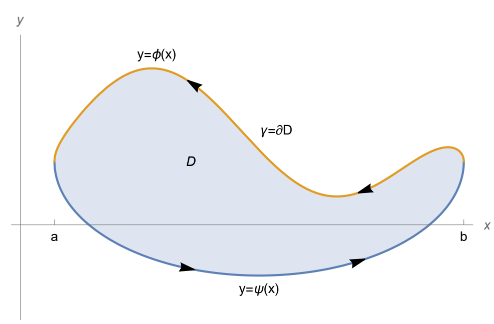
$$\oint_\gamma \vec{F} = \int_D (\partial_y F_1 - \partial_x F_2)\, dxdy. \tag{4.6.1}$$

Proof. For the proof, we assume, for simplicity, that the domain D is the region delimited by two functions, that is,

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b],\ \phi(x) \leqslant y \leqslant \psi(y)\}$$

and $\phi(a) = \psi(a)$, $\phi(b) = \psi(b)$. In this case

$$\partial D = \mathrm{Graph}(\phi) \cup \mathrm{Graph}(\psi).$$



Since $\gamma = \partial\Omega$, we may assume the following parameterization for γ:

$$\gamma = \gamma_\psi + \gamma_\phi, \quad \text{where } \gamma_\psi(x) = (x, \psi(x)),\ x \in [a, b],\ \ \gamma_\phi(x) = (x, \phi(x)),\ x \in [b, a].$$

With $x \in [b, a]$ we mean that x runs from b to a (right to left). With all these premises,

$$\begin{aligned} \oint_\gamma \vec{F} &= \int_a^b \vec{F}(x, \psi(x)) \cdot (1, \psi'(x))\, dx + \int_b^a \vec{F}(x, \phi(x)) \cdot (1, \phi'(x))\, dx \\ &= \int_a^b F_1(x, \psi(x)) + F_2(x, \psi(x))\psi'(x)\, dx - \int_a^b F_1(x, \phi(x)) + F_2(x, \phi(x))\phi'(x)\, dx \\ &= -\int_a^b (F_1(x, \phi(x)) - F_1(x, \psi(x)))\ dx - \int_a^b (F_2(x, \phi(x))\phi'(x) - F_2(x, \psi(x))\psi'(x))\ dx. \end{aligned}$$

Now, by the fundamental theorem of integral calculus,

$$F_1(x, \phi(x)) - F_1(x, \psi(x)) = \int_{\psi(x)}^{\phi(x)} \partial_y F_1(x, y)\, dy.$$

A bit more complicate the remaining term. First notice that, according to the fundamental theorem of integral calculus,

$$\partial_x \int_{\psi(x)}^{\phi(x)} F_2(x,y)\,dy = F_2(x,\phi(x))\phi'(x) - F_2(x,\psi(x))\psi'(x) + \int_{\psi(x)}^{\phi(x)} \partial_x F_2(x,y)\,dy$$

thus,

$$\begin{aligned}\int_a^b \left(F_2(x,\phi(x))\phi'(x) - F_2(x,\psi(x))\psi'(x)\right)\,dx &= \int_a^b \left(\partial_x \int_{\psi(x)}^{\phi(x)} F_2(x,y)\,dy\right)\,dx - \int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x F_2(x,y)\,dy\right)\,dx \\ &= \left[\int_{\psi(x)}^{\phi(x)} F_2(x,y)\,dy\right]_{x=a}^{x=b} - \int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x F_2(x,y)\,dy\right)\,dx \\ &= -\int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x F_2(x,y)\,dy\right)\,dx\end{aligned}$$

being $\phi(a) = \psi(a)$, $\phi(b) = \psi(b)$. In conclusion,

$$\oint_\gamma \vec{F} = -\int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_y F_1(x,y) + \partial_x F_2(x,y)\,dy\right)\,dx = \int_D \left(\partial_y F_1 - \partial_x F_2\right)\,dxdy,$$

which is the conclusion. □

In particular, we have the

Corollary 4.6.2

Let $\vec{F}$ be an irrotational field on Ω and suppose that for every $\gamma \subset \Omega$ circuit, it holds $\gamma = \partial D$ where $D \subset \Omega$ open. Then $\vec{F}$ is conservative.

Proof. Let $\gamma \subset \Omega$ be a closed path. According to hypotheses, $\gamma = \partial D$ for some open $D \subset \Omega$. But then,

$$\oint_\gamma \vec{F} = \int_D (\partial_y F_1 - \partial_x F_2)\,dxdy = 0.$$

The conclusion follows now from Theorem 3.3. □

Green formula has some other curious consequence:

Corollary 4.6.3: Area formula

Let D be an open and bounded domain with $\partial D = \gamma$, where $\gamma = (x,y)$ is a counterclockwise oriented circuit. Then

$$\lambda_2(D) = \oint_\gamma y\,dx = -\oint_\gamma x\,dy.$$

Proof. Let $\vec{F} = (y,0)$. Then $\partial_y F_1 - \partial_x F_2 = 1$, thus

$$\oint_\gamma (y,0) = \int_D 1\,dxdy = \lambda_2(D).$$

Example 4.6.4. *Compute the area of a disk of radius r.*

Sol. — Let $\gamma(t) = r(\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\lambda_2(D) = \int_0^{2\pi} r \sin t \; d(r\cos t) = -r^2 \int_0^{2\pi} (\sin t)^2 \; dt.$$

Integrating by parts

$$-\int_0^{2\pi} (\sin t)^2 \; dt = [\sin t \cos t]_{t=0}^{t=2\pi} - \int_0^{2\pi} (\cos t)^2 \; dt = 2\pi - \int_0^{2\pi} (\sin t)^2 \; dt,$$

from which $-\int_0^{2\pi} (\sin t)^2 \; dt = -\pi$. Therefore $\lambda_2(D) = \pi r^2$ as well known. □

4.7. Exercises

Exercise 4.7.1. Compute

1. $\displaystyle\int_{0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant 1-y^2} xe^y \; dxdy.$ 2. $\displaystyle\int_{0\leqslant y\leqslant 1-x^2} \frac{x}{2+y} \; dxdy.$ 3. $\displaystyle\int_{|y|\leqslant 1-x^2} \frac{1}{1+y} \; dxdy.$

4. $\displaystyle\int_{[0,1]\times[2,4]} \frac{1}{(x-y)^2} \; dxdy$ 5. $\displaystyle\int_{1\leqslant x\leqslant 2,\ \frac{1}{x}\leqslant y\leqslant x} \frac{x}{y} \; dxdy.$ 6. $\displaystyle\int_{[0,1]^2} e^{\max\{x^2,y^2\}} \; dxdy.$

7. $\displaystyle\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4} \; dxdy$ 8. $\displaystyle\int_{0\leqslant x\leqslant y\leqslant 1} x\sqrt{y^2-x^2} \; dxdy.$ 9. $\displaystyle\int_{|xy|\leqslant 1} \frac{x^2e^{-x^2}}{1+(xy)^2} \; dxdy.$

Exercise 4.7.2. Compute

1. $\displaystyle\int_{[1,+\infty[^3} y^3z^8e^{-xy^2z^3} \; dxdydz.$ 2. $\displaystyle\int_{x\geqslant 0,\ y\geqslant 0,\ x+y+z\leqslant 1} xyz \; dxdydz.$ 3. $\displaystyle\int_{0\leqslant x,y\leqslant 1,\ 0\leqslant z\leqslant x^2} zy^2\sqrt{x^2+zy} \; dxdydz.$

Exercise 4.7.3. Let $D := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant r^2\}$. Draw D and describe it in polar coords. Determine its barycenter and compute the integral

$$\int_D \frac{x+y}{x^2+y^2} \; dxdy.$$

Exercise 4.7.4 (polar, spherical, cylindrical coords). Compute the volume of

1. $\left\{(x, y, z) : 9(1-\sqrt{x^2+y^2})^2 + 4z^2 \leqslant 1\right\}.$ 2. $\left\{(x, y, z) \ :\ x^2+y^2+z^2 \leqslant r^2,\ \left(x-\frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}\right\}.$

3. $\left\{(x, y, z) \in \mathbb{R}^3 \ :\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\right\}$, $(a, b, c > 0)$. 4. $\left\{(x, y, z) \ :\ x^2+y^2 \leqslant 1,\ x^2+z^2 \leqslant 1,\ y^2+z^2 \leqslant 1\right\}.$

5. $\left\{(x, y, z) \in \mathbb{R}^3 \ :\ x^2+y^2 \leqslant 4,\ 4x^2+4y^2+z^2 \leqslant 64\right\}.$ 6. $\left\{(x, y, z) \in \mathbb{R}^3 \ :\ x^2+y^2+z^2 \leqslant 16,\ x^2+y^2 \geqslant 4\right\}$

7. $\left\{(x, y, z) \in \mathbb{R}^3 \ :\ z \geqslant \sqrt{x^2+y^2},\ x^2+y^2+z^2 \leqslant 1\right\}.$ 8. $\left\{(x, y, z) \in \mathbb{R}^3 \ :\ z \geqslant x^2+y^2,\ z \leqslant 18-x^2-y^2\right\}.$

EXERCISE 4.7.5. Compute

1. $\displaystyle\int_{x^2+y^2\leqslant 4}\sqrt{4-x^2-y^2}\,dxdy$
2. $\displaystyle\int_{x^2+y^2\leqslant 1}\frac{1}{1+x^2+y^2}\,dxdy.$
3. $\displaystyle\int_{\mathbb{R}^2} e^{-(x^2+2y^2)}\,dxdy.$
4. $\displaystyle\int_{x^2+y^2\leqslant 16,\ -5\leqslant z\leqslant 4}\sqrt{x^2+y^2}\,dxdydz.$
5. $\displaystyle\int_{\mathbb{R}^3}\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,dxdydz.$
6. $\displaystyle\int_{[0,+\infty[^3}\frac{x}{1+(x^2+2y^2+3z^2)^2}\,dxdydz.$

EXERCISE 4.7.6. By using the suggested change of variables, compute

1. $\displaystyle\int_D xy\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 3,\ x\leqslant y\leqslant 3x\},\ (u=xy,\ v=\frac{y}{x}).$
2. $\displaystyle\int_D y^2\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 2,\ 1\leqslant xy^2\leqslant 2\}.\ (u=xy,\ v=xy^2).$
3. $\displaystyle\int_D \sqrt{x^2-y^2}\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant x^2-y^2\leqslant 2,\ x\leqslant y\leqslant 2x\}.\ (u=x^2-y^2,\ v=\frac{y}{x}).$

EXERCISE 4.7.7. Let $a>1$ and

$$D_a:=\left\{(x,y)\in\mathbb{R}^2\ :\ \frac{1}{ax}\leqslant y\leqslant\frac{1}{x},\ x^2\leqslant y\leqslant ax^2\right\}.$$

Draw D_a. Then, using the change of variables $u=xy$, $v=\frac{y}{x}$, compute

$$I(a):=\int_{E_a}\frac{x^2}{y}e^{xy}\,dxdy.$$

EXERCISE 4.7.8. Let

$$f(x,y):=\frac{x^{3/2}}{\sqrt{y-x}}e^{-(xy)^{3/2}},\quad (x,y)\in D:=\{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant x\leqslant y\}.$$

Use the change of variables $(u,v):=(xy,x/y)$ to compute $\int_D f$.

EXERCISE 4.7.9. Let

$$f(x,y):=\frac{\log(xy)}{y(x+y^2)^7},\ (x,y)\in D:=\{(x,y)\in[0,+\infty[^2:\ xy\geqslant 1\}.$$

Use the change of variables $(u,v):=(xy,\frac{y^2}{x})$ to compute $\int_D f$.

CHAPTER 5

Surface integrals

Consider a surface $\mathscr{M}$ embedded into Euclidean space $\mathbb{R}^3$. Intuitively, a surface is a two dimensional object in space, like a curved sheet of paper. It seems natural that as such, $\mathscr{M}$ will have measure zero in $\mathbb{R}^3$, that is $\lambda_3(\mathscr{M})$. This because λ_3 is a volume, and a surface should have positive area while its volume is zero. The problem we tackle here is how to define a proper concept of *area*. Of course, we cannot use $\lambda_2(\mathscr{M})$ because λ_2 applies to sets contained in $\mathbb{R}^2$. Furthermore, we may expect that *curvature of $\mathscr{M}$* matters.

The concept of area of a surface embedded in $\mathbb{R}^3$ is just, as we will see, a particular case of the concept of *surface integral*, an operation we will introduce in this Chapter. Through this, many important entities used in Physics and Engineering, as *fluxes of a vector fields through a surface*, can be rigorously defined and computed. This shows, once more, how much important is Integration in applications of Mathematics.

5.1. Parametric Surfaces

In this section we will give a definition to the concept of *area of a surface embedded into an Euclidean space* $\mathbb{R}^3$. The first step is to give a precise definition of what is a surface. Normally, there are two ways to define a surface in the space:

- through a *Cartesian equation*;
- through a *parametrization*.

The first method is perhaps the most simple and general. Let us see some examples:

Example 5.1.1. The following are surfaces embedded in $\mathbb{R}^3$:

- a *plane* $ax + by + cz = r$ where $a, b, c, r \in \mathbb{R}$, and $(a, b, c) \neq \vec{0}$;
- a *spherical surface*, as for example
$$x^2 + y^2 + z^2 = r^2.$$
- a *paraboloid*,
$$z = ax^2 + by^2,\ a, b > 0.$$
- a *cone*,
$$z^2 = ax^2, by^2,\ a, b > 0.$$

If a Cartesian equation is a natural way to describe an embedded surface, also parametrizations a pretty natural. Everybody, even a kid, knows a remarkable example of this: our planet. Indeed, nobody uses Cartesian coordinates to identify points on the surface of the planet. Rather, we use *latitude* and *longitude*, which are, in fact, *spherical coordinates*:

Example 5.1.2. Let $\mathbb{S}_r := \{(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 = r^2\}$ be the spherical surface of radius $r > 0$. Points $(x, y, z) \in \mathbb{S}_r$ can be also described through
$$(x, y, z) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\,,$$

where $(\theta, \varphi) \in [0, 2\pi] \times [0, \pi]$. Referring $\theta = 0$ to the Greenwich meridian, φ is the so-called *colatitude*, that is *latitude* $= \theta - \frac{\pi}{2}$. ◻

We formalize now the idea presented in the previous example. We had a set $\mathscr{M} \subset \mathbb{R}^3$ such that

$$\mathscr{M} = \Phi(D), \text{ where } \Phi = \Phi(u, v) \ : \ D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$$

We may expect that some technical condition is needed to ensure $\mathscr{M}$ be a true surface, that is a *two dimensional object*. Indeed, if, for example, Φ is a constant function, say $\Phi(u, v) = (0, 0, 0)$ for all $(u, v) \in D = \mathbb{R}^3$, clearly

$$\Phi(D) = \{\vec{0}\}$$

is not a surface. So, what is the right condition to ask on Φ in order $\mathscr{M} = \Phi(D)$ be a true surface?

The idea is suggested from the example of coordinates on Earth. Let us consider the standard spherical parametrization

$$\Phi(\theta, \varphi) := (r \cos\theta \sin\varphi, r \sin\theta \sin\varphi, r \cos\varphi) .$$

We may notice, if we fix the longitude $\theta = \theta_0$, the line

$$\varphi \longmapsto \Phi(\theta_0, \varphi),$$

is a meridian while, for a fixed co-latitude $\varphi = \varphi_0$, the line

$$\theta \longmapsto \Phi(\theta, \varphi_0),$$

is a parallel. Latitude and longitude are good coordinates on the planet because locally, around to a point (θ_0, φ_0), they works are cartesian coordinates x, y on the plane xy. In other words, the longitude axis $\varphi = \varphi_0$ and latitude axis $\theta = \theta_0$ are *perpendicular*. How this geometrical intuition might be expressed known that lines $\varphi = \varphi_0$ and $\theta = \theta_0$ are circles on $\mathbb{S}_r$? Taking the parallel $\varphi = \varphi_0$, that is the curve $\theta \longmapsto \Phi(\theta, \varphi_0)$, its tangent vector at $theta = \theta_0$ is

$$\partial_\theta(\theta_0, \varphi_0).$$

Similarly, the tangent vector to the meridian $\theta = \theta_0$ at $\varphi = \varphi_0$ is

$$\partial_\varphi\Phi(\theta_0, \varphi_0).$$

These two vectors are perpendicular. Indeed:

$$\partial_\theta\Phi(\theta, \varphi) = (-r \sin\theta \sin\varphi, r \cos\theta \sin\varphi, 0), \ \ \partial_\varphi(\Phi(\theta, \varphi) = (r \cos\theta \cos\varphi, r \sin\theta \cos\varphi, -r \sin\varphi),$$

thus

$$\partial_\theta\Phi \cdot \partial_\varphi\Phi = -r^2(\sin\theta \sin\varphi)(\cos\theta \cos\varphi) + r^2(\cos\theta \sin\varphi)(\sin\theta \cos\varphi) \equiv 0.$$

This condition might be used in general as condition $\Phi = \Phi(u, v)$ must obey in order $\mathscr{M} = \Phi(D)$ be a surface. In this way, u, v would act as "orthogonal coordinates" on $\mathscr{M}$. Written in this form, this condition is too restrictive. We actually just need that u and v axes be *linearly independent*. This is why we introduce the

Definition 5.1.3

Let $\Phi = \Phi(u, v) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. We say that Φ is an **immersion** if

$$\partial_u\Phi(u, v), \ \partial_v\Phi(u, v) \text{ are linearly independent } \forall (u, v) \in D.$$

The set $\mathscr{M} := \Phi(D)$ is called **parametric surface described by** Φ and Φ is called **parametrization** of $\mathscr{M}$.

Remark 5.1.4. Linear independence of $\partial_u \Phi, \partial_v \Phi$ may be equivalently expressed in the following way:

$$\operatorname{rank} \begin{bmatrix} \partial_u \Phi \\ \\ \partial_v \Phi \end{bmatrix} = 2. \quad \square$$

Let us review some of the above examples.

Example 5.1.5. *A cartesian plane $ax + by + cz = r$, with $(a, b, c) \neq \vec{0}$, is a parametric surface.*

Sol. — Assume, for example $c \neq 0$. Then

$$ax + by + cz = r, \quad \Longleftrightarrow \quad z = \frac{1}{c}(r - ax - by).$$

Define

$$\Phi = \Phi(x, y) := \left(x, y, \frac{1}{c}(r - ax - by)\right), \ (x, y) \in D = \mathbb{R}^2.$$

Then Φ is a parametrization of $\mathscr{M} := \{(x, y, z) \ : ax + by + cz = r\}$. Indeed:

$$\operatorname{rank} \begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & -\frac{a}{c} \\ \\ 0 & 1 & -\frac{b}{c} \end{bmatrix} = 2,$$

clearly. Thus Φ is an immersion and by construction $\mathscr{M} = \Phi(D)$. $\square$

Example 5.1.6. *Let $\mathscr{M} := \{(x, y, z) \ : \ z = ax^2 + by^2\}$, $a, b > 0$ fixed. Then $\mathscr{M}$ is a parametric surface.*

Sol. — Indeed, define

$$\Phi = \Phi(x, y) = (x, y, ax^2 + by^2), \ (x, y) \in \mathbb{R}^2 =: D.$$

Then

$$\operatorname{rank} \begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 2ax \\ \\ 0 & 1 & 2by \end{bmatrix} = 2,$$

clearly. Thus Φ is an immersion and by construction $\mathscr{M} = \Phi(D)$. $\square$

Previous examples suggests a general fact:

Proposition 5.1.7

Graphs of regular functions are parametric surfaces. Precisely, if $f = f(x, y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ then

$$\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z = f(x, y)\right\},$$

is a parametric surface with standard parametrization $\Phi(x, y) := (x, y, f(x, y))$.

Proof. It is sufficient to notice that

$$\operatorname{rank} \begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & \partial_x f \\ \\ 0 & 1 & \partial_y f \end{bmatrix} = 2. \quad \square$$

5.2. Area of a parametric surface

Now we are acquainted with the definition of *parametric surface*, let us see how to use this to define a proper concept of *area for* $\mathscr{M} = \Phi(D)$, where Φ is an immersion on D.

Surf0.pdf Surf2.pdf Surf1.jpg

A natural idea consists in dividing D in small rectangles $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$. It seems then reasonable that

$$\mathrm{Area}(\mathscr{M}) = \sum_{i,j} \mathrm{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right).$$

Now, if $u_i \sim u_{i+1}$ and $v_j \sim v_{j+1}$ we could say that $\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])$ is almost a parallelogram of sides $\Phi(u_i, v_{j+1}) - \Phi(u_i, v_j)$ and $\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j)$.

Surf3.pdf Surf4.pdf areaparall.pdf

Then,

$$\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j) \approx \partial_u \Phi(u_i, v_j)(u_{i+1} - u_i), \quad \Phi(u_i, v_{j+1}) - \Phi(u_i, v_j) \approx \partial_v \Phi(u_i, v_j)(v_{j+1} - v_j),$$

so

$$\mathrm{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right) \approx \mathrm{Area}\left([\partial_u \Phi(u_i, v_j)\ \partial_v \Phi(u_i, v_j)]\right)(u_{i+1} - u_i)(v_{j+1} - v_j),$$

where we settled $[\vec{a}\ \vec{b}]$ the parallelogram of sides $\vec{a}$ and $\vec{b}$. Notice incidentally that $\mathrm{Area}[\vec{a}, \vec{b}] \neq 0$ iff $\vec{a}$ and $\vec{b}$ are linearly independent, which is true here because Φ is supposed to be an immersion. Elementary geometry gives the formula

$$\mathrm{Area}[\vec{a}, \vec{b}] = \|\vec{a}\| \|\vec{b}\| \sin\theta,$$

where θ is the angle formed by $\vec{a}$ and $\vec{b}$. By recalling the vector product,

$$\mathrm{Area}[\vec{a}, \vec{b}] = \|\vec{a}\| \|\vec{b}\| \sin\theta = \|\vec{a} \times \vec{b}\|, \quad \text{where } \vec{a} \times \vec{b} = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Returning to the initial problem

$$\mathrm{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right) \approx \|\partial_u \Phi(u_i, v_i) \times \partial_v \Phi(u_i, v_i)\| (u_{i+1} - u_i)(v_{j+i} - v_j)$$

hence

$$\mathrm{Area}(\mathscr{M}) \approx \sum_{i,j} \|\partial_u \Phi(u_i, v_i) \times \partial_v \Phi(u_i, v_i)\| (u_{i+1} - u_i)(v_{j+i} - v_j) \approx \int_D \|\partial_u \Phi(u, v) \times \partial_v \Phi(u, v)\|\, du dv.$$

This informal argument justifies the

Definition 5.2.1

Let $\mathscr{M} = \Phi(D)$ be a parametric surface. We call **area of** $\mathscr{M}$ the quantity

$$\sigma_2(\mathscr{M}) := \int_D \|\partial_u \Phi(u, v) \times \partial_v \Phi(u, v)\|\, du dv \equiv \int_D \|\partial_u \Phi \times \partial_v \Phi\|. \tag{5.2.1}$$

EXAMPLE 5.2.2. *Compute the area of a spherical surface of radius* r, $\mathbb{S}^2_r := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 = r^2\}$.

SOL. — Recalling the natural parametrization of the spherical surface,

$$\Phi(\theta,\varphi) := r\,(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)\,,\ (\theta,\varphi) \in [0,2\pi]\times[0,\pi],$$

we have

$$\partial_\theta\Phi = r(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0),\quad \partial_\varphi\Phi = r(\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi).$$

Shortening $S = \sin\theta$, $C = \cos\theta$, $s = \sin\varphi$, $c = \cos\varphi$,

$$\partial_\theta\Phi \times \partial_\varphi\Phi = \det\begin{bmatrix} i & j & k \\ -rSs & rCs & 0 \\ rCc & rSc & -rs \end{bmatrix} = r^2(-Cs^2, -Ss^2, -S^2sc - C^2sc) = -r^2(Cs^2, Ss^2, sc),$$

thus

$$\|\partial_\theta\Phi \times \partial_\varphi\Phi\| = r^2\sqrt{C^2s^4 + S^2s^4 + s^2c^2} = r^2\sqrt{s^2(s^2+c^2)} = r^2\sqrt{s^2} = r^2|s| = r^2|\sin\varphi|.$$

Therefore

$$\sigma_2(\mathbb{S}^2_r) = \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} r^2|\sin\varphi|\ d\theta d\varphi = 2\pi r^2\int_0^\pi \sin\varphi\ d\varphi = 4\pi r^2. \qquad \square$$

Apparently, formula (5.2.1) depends on the parametrization Φ. Were this be the case, this would be disturbing. Fortunately, it may be proved that, under natural assumptions, $\sigma_2(\mathscr{M})$ does not depend on a specific parametrization. We will deduce as a particular case of invariance of a surface integral.

5.3. Surface Integral

We may see at the area of a parametric surface as

$$\sigma_2(\mathscr{M}) = \int_{\mathscr{M}} 1\ d\sigma_2,$$

where

$$\boxed{d\sigma_2 = \|\partial_u\Phi(u,v) \times \partial_v\Phi(u,v)\|\ dudv,}$$

is called **area element**. This leads to a natural extension:

Definition 5.3.1

Let $f : \mathscr{M} \longrightarrow \mathbb{R}$ be continuous on $\mathscr{M} = \Phi(D)$ parametric surface. We set

$$\int_{\mathscr{M}} f\ d\sigma_2 := \int_D f(\Phi(u,v))\|\partial_u\Phi(u,v) \times \partial_v\Phi(u,v)\|\ dudv \equiv \int_D f(\Phi)\|\partial_u\Phi \times \partial_v\Phi\|, \tag{5.3.1}$$

provided integral exists and it is finite.

Apparently, the surface integral $\int_{\mathscr{M}} f\ d\sigma$ depends on the specific parametrization Φ. In particular, if $f \equiv 1$, area of a parametric surface depends on the parametrization. Of course, this is not the case if we can pass from one parametrization to another one regularly:

Proposition 5.3.2

Let $\mathscr{M} = \Phi(D) = \Psi(\widetilde{D})$, with $\Phi, \Psi \in \mathscr{C}^1$ immersions such that $\Phi^{-1} \circ \Psi$ be a diffeomorphism (that is, it is differentiable with its inverse). Then

$$\int_D f(\Phi(u,v))\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|\, dudv = \int_{\widetilde{D}} f(\Psi(\xi,\eta))\|\partial_\xi \Psi(\xi,\eta) \times \partial_\eta \Psi(\xi,\eta)\|\, d\xi d\eta \tag{5.3.2}$$

Proof. Call $x = (u,v)$ and $y = (\xi,\eta)$ and set $\Gamma := \Phi^{-1} \circ \Psi$ be the change of parametrization.

[illegible]

Then, by chain rule,

$$\Psi(y) = \Phi(\Phi^{-1}(\Psi(y)) = \Phi(\Gamma(y)), \implies \Psi'(y) = \Phi'(\Gamma(y))\Gamma'(y),$$

that is

$$[\partial_\xi \Psi\ \partial_\eta \Psi] = [\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\Gamma'.$$

Therefore

$$\begin{aligned}\|\partial_\xi \Psi \times \partial_\eta \Psi\| &= \mathrm{Area}[\partial_\xi \Psi\ \partial_\eta \Psi] = \mathrm{Area}\left([\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\Gamma'\right) = \mathrm{Area}\left([\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\right) |\det \Gamma'(y)| \\ &= \|\partial_u \Phi(\Gamma) \times \partial_v \Phi(\Gamma)\| |\det \Gamma'|.\end{aligned}$$

To finish, by the change of variables

$$\begin{aligned}\int_{\widetilde{D}} f(\Psi(y))\|\partial_\xi(y)\Psi \times \partial_\eta \Psi(y)\|\, dy &= \int_{\widetilde{D}} f(\Phi(\Gamma(y)))\|\partial_u \Phi(\Gamma(y)) \times \partial_v \Phi(\Gamma(y))\| |\det \Gamma'(y)|\, dy \\ &\overset{x=\Gamma(y)}{=} \int_D f(\Phi(x))\, \|\partial_u \Phi(x) \times \partial_v \Phi(x)\|\, dx. \quad \square\end{aligned}$$

Similarly to the case of multidimensional integral, the **barycentre** of a surface $\mathscr{M}$ is the point

$$(\overline{x}, \overline{y}, \overline{z}) = \frac{1}{\sigma_2(\mathscr{M})}\left(\int_{\mathscr{M}} x\, d\sigma_2, \int_{\mathscr{M}} y\, d\sigma_2, \int_{\mathscr{M}} z\, d\sigma_2\right).$$

If $\varrho(x,y,z)$ is a mass density on $\mathscr{M}$, the **total mass** of $\mathscr{M}$ is

$$\mu(\mathscr{M}) := \int_{\mathscr{M}} \varrho\, d\sigma_2.$$

If the distribution is homogeneous, that is if $\varrho \equiv \varrho_0$, then $\mu(\mathscr{M}) = \varrho_0 \sigma_2(\mathscr{M})$. The **center of mass** of $\mathscr{M}$ is the point

$$\frac{1}{\mu(\mathscr{M})}\left(\int_{\mathscr{M}} x\varrho\, d\sigma_2, \int_{\mathscr{M}} y\varrho\, d\sigma_2, \int_{\mathscr{M}} z\varrho\, d\sigma_2\right).$$

For homogeneous bodies the center of mass is the barycenter.

An important particular case is the following:

Proposition 5.3.3

Let $\varphi = \varphi(x, y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class $\mathscr{C}^1$ and let

$$\mathscr{M} := \{(x, y, \varphi(x, y)) \ : \ (x, y) \in D\}.$$

Then

$$\int_{\mathscr{M}} f \, d\sigma_2 = \int_D f(x, y, g(x, y))) \sqrt{1 + \|\nabla\varphi(x, y)\|^2} \, dxdy. \tag{5.3.3}$$

In particular, we have the **area of a graph formula**,

$$\sigma_2(\mathscr{M}) = \int_D \sqrt{1 + \|\nabla\varphi\|^2} \, dxdy. \tag{5.3.4}$$

Proof. Just compute the area element: here $\Phi : D \longrightarrow \mathbb{R}^3$ is given by $\Phi(x, y) = (x, y, \varphi(x, y))$. Then

$$\partial_u\Phi = (1, 0, \partial_x\varphi), \ \partial_v\Phi = (0, 1, \partial_y\varphi), \ \Longrightarrow \ \partial_x\Phi \times \partial_y\Phi = \det\begin{bmatrix} i & j & k \\ 1 & 0 & \partial_x\varphi \\ 0 & 1 & \partial_y\varphi \end{bmatrix} = (-\partial_x\varphi, -\partial_y\varphi, 1),$$

so

$$\|\partial_x\Phi \times \partial_y\Phi\| = \sqrt{(\partial_x\varphi)^2 + (\partial_y\varphi)^2 + 1} = \sqrt{1 + \|\nabla\varphi\|^2}. \qquad \square$$

Example 5.3.4. *Compute, by using area of graph formula, the area of spherical surface $\mathbb{S}^2_r$.*

Sol. — We can see the spherical surface $\mathbb{S}^2_r$ as twice the area of an half sphere. This last is a graph: for instance in the half plane $z \geqslant 0$, $z = \sqrt{r^2 - (x^2 + y^2)} =: \varphi(x, y)$. Then

$$\sigma_2(\mathbb{S}^2_r) = 2\int_{x^2+y^2\leqslant r^2} \sqrt{1 + \|\nabla\varphi(x, y)\|^2} \, dxdy.$$

Being $\nabla\varphi = -\frac{(x,y)}{\sqrt{r^2-(x^2+y^2)}}$ we have

$$\begin{aligned} \sigma_2(\mathbb{S}^2(r)) &= 2\int_{x^2+y^2\leqslant r^2} \sqrt{1 + \frac{x^2+y^2}{r^2-(x^2+y^2)}} \, dxdy = 2\int_{0\leqslant\rho\leqslant r, \ 0\leqslant\theta\leqslant 2\pi} \sqrt{1 + \frac{\rho^2}{r^2-\rho^2}}\rho \, d\rho d\theta \\ &= 4\pi r \int_0^r \frac{\rho}{\sqrt{r^2-\rho^2}} \, d\rho = 4\pi r \left[-\sqrt{r^2-\rho^2}\right]_{\rho=0}^{\rho=r} = 4\pi r^2. \qquad \square \end{aligned}$$

Example 5.3.5. *Compute the barycenter of an half sphere.*

Sol. — The half sphere is the graph of $\beta : D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant r^2\} \longrightarrow \mathbb{R}$, $\varphi(x, y) = \sqrt{r^2 - (x^2 + y^2)}$. By simmetry it is evident that $\int_{\mathscr{M}} x \, d\sigma_2 = \int_{\mathscr{M}} y \, d\sigma_2 = 0$. It remains

$$\int_{\mathscr{M}} z \, d\sigma_2 = \int_{x^2+y^2\leqslant r^2} z\frac{r}{\sqrt{r^2-(x^2+y^2)}} \, dxdy = r\int_{x^2+y^2\leqslant r^2} dxdy = \pi r^3.$$

Being $\sigma_2(\mathscr{M}) = 2\pi r^2$ we obtain the barycenter as the point $\frac{1}{2\pi r^2}\left(0, 0, \pi r^3\right) = \left(0, 0, \frac{r}{2}\right)$. $\square$

5.3.1. Guldino's formula. This is an analogous for surfaces of the thm of Pappo. Let's consider a rotation surface $\mathscr{M}$ obtained by a rotation of a curve respect to one of the axes. To fix ideas let $\gamma = (y(t), z(t)) \in \mathscr{C}^1([a, b])$ be in the plan yz with $y(t) > 0$.

SurfG1.pdf SurfG2.jpg

We may describe the rotation around the z axis as

$$\mathscr{M} = \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \ : \ t \in [a, b],\ \theta \in [0, 2\pi]\} = \Phi([a, b] \times [0, 2\pi]),$$

where of course $\Phi(t, \theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$. Let's prove the

Theorem 5.3.6: Guldino

Let $\gamma(t) = (y(t), z(t)) \in \mathscr{C}^1([a, b])$ with $\|\gamma'\| \neq 0$ and let

$$\mathscr{M} := \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \ : \ t \in [a, b],\ \theta \in [0, 2\pi]\},$$

be the rotation surface of γ around the z axis. Then

$$\sigma_2(\mathscr{M}) = 2\pi \int_a^b y(t)\|\gamma'(t)\|\, dt. \tag{5.3.5}$$

Proof. Notice that $\mathscr{M} = \Phi([a, b] \times [0, 2\pi])$ where $\Phi(t, \theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$, hence

$$\partial_t \Phi = (y'\cos\theta, y'\sin\theta, z'),\ \ \partial_\theta \Phi = (-y\sin\theta, y\cos\theta, 0),$$

hence the area element is

$$\begin{aligned} \|\partial_t\Phi \times \partial_\theta\Phi\| &= \left\| \det \begin{bmatrix} i & j & k \\ y'\cos\theta & y'\sin\theta & z' \\ -y\sin\theta & y\cos\theta & 0 \end{bmatrix} \right\| = \|(yz'\cos\theta, yz'\sin\theta, yy'(\cos^2\theta + \sin^2\theta))\| \\ &= \sqrt{y^2 z'^2(\cos^2\theta + \sin^2\theta) + y^2 y'^2} = \sqrt{y^2(y'^2 + z'^2)} \overset{y \geqslant 0}{=} y\|\gamma'\|. \end{aligned}$$

Finally

$$\sigma_2(\mathscr{M}) = \int_{t\in[a,b],\ \theta\in[0,2\pi]} y\|\gamma'\|\, dt d\theta = 2\pi \int_a^b y\|\gamma'\|\, dt. \quad \square$$

Example 5.3.7. *Compute the area of a torus $\mathbb{T}_{R,r}$ of radius $0 < r < R$.*

Sol. — The torus may be seen as the rotation of a circumference of radius r at distance R by the z axis. Taking the section in the plane yz with $y \geqslant 0$,

$$\gamma \ : \ (y(t), z(t)) = (R + r\cos t, r\sin t),\ t \in [0, 2\pi].$$

By Guldino's formula

$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi \int_0^{2\pi} (R + r\cos t)\|\gamma'\|\, dt.$$

Now, $\gamma' = (-r\sin t, r\cos t)$ hence clearly $\|\gamma'\| = r$. Therefore

$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi \int_0^{2\pi} (R + r\cos t) r \, dt = 4\pi^2 rR. \quad \square$$

5.4. Flux of a vector field

One of the main applications of surface integrals is to give a correct definition of *flux of a vector field through a surface*. Let $\mathscr{M} = \Phi(D)$ be a parametric surface and let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a *vector field* defined on $\Omega \supset \mathscr{M}$. Our aim is to define

$$\int_{\mathscr{M}} \vec{F} \cdot \vec{n} \, d\sigma_2, \quad \text{where } \vec{n}(\vec{x}) = \text{unit normal to } \mathscr{M} \text{ at point } \vec{x}. \tag{5.4.1}$$

Of course, the main problem is how to define the normal$\vec{n}(\vec{x})$. Assume that $\vec{x} = \Phi(u,v)$. Then, vectors $\partial_u \Phi(u,v)$ and $\partial_v \Phi(u,v)$ are tangent to $\mathscr{M}$.

SurfA3.pdf

It is therefore natural to pose

$$\vec{n}_\Phi(u,v) := \frac{\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)}{\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|}. \tag{5.4.2}$$

Notice that also $-\vec{n}$ is unitary and orthogonal to $\mathscr{M}$. To obtain this as normal it is sufficient to change the order of the two parameters. Indeed, by setting $\Psi(u,v) := \Phi(v,u)$, then

$$\partial_u \Psi = \partial_v \Phi, \quad \partial_v \Psi = \partial_u \Phi, \implies \partial_u \Psi \times \partial_v \Psi = \partial_v \Phi \times \partial_u \Phi = -\partial_u \Phi \times \partial_v \Phi,$$

so

$$\vec{n}_\Psi = -\vec{n}_\Phi.$$

This remark shows that there could be an ambiguity in the notation (5.4.1): indeed the surface integral doesn't depend by the specific parametrization, but the vector $\vec{n}$ depends clearly by this in such a way that we could have different parametrization of the same $\mathscr{M}$ and fluxes with opposite signs. This doesn't seems a dramatic problem but some care is needed when we define the flux:

Definition 5.4.1

Let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a continuous vector field on $\Omega \supset \mathscr{M} = \Phi(D)$ parametric surface. We call **flux of $\vec{F}$ through $\mathscr{M}$**

$$\langle \vec{F} \rangle_{\mathscr{M}} := \int_{\mathscr{M}} \vec{F} \cdot \vec{n}_\Phi \, d\sigma_2. \tag{5.4.3}$$

Notice that

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F}(\Phi) \cdot \frac{\partial_u \Phi \times \partial_v \Phi}{\|\partial_u \Phi \times \partial_v \Phi\|} \|\partial_u \Phi \times \partial_v \Phi\| \, dudv = \int_D \vec{F}(\Phi) \cdot (\partial_u \Phi \times \partial_v \Phi) \, dudv.$$

It is easy to check that

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \det\left[\vec{F}(\Phi(u,v)) \; \partial_u \Phi(u,v) \; \partial_v \Phi(u,v) \right] \, dudv. \tag{5.4.4}$$

Example 5.4.2. *Compute the flux of $\vec{F} = (x, y, z)$ through the paraboloid $z = x^2 + y^2$ included between $z = 1$ and $z = 2$.*

Sol. — On a $\mathscr{M} = \Phi(D)$ où $\Phi(u, v) = (u, v, u^2 + v^2)$ et $(u, v) \in D = \{1 \leqslant u^2 + v^2 \leqslant 2\}$. Par la (5.4.4)

$$\begin{aligned}\langle \vec{F} \rangle_{\mathscr{M}} &= \int_{1\leqslant u^2+v^2\leqslant 2} \det \begin{bmatrix} u & 1 & 0 \\ v & 0 & 1 \\ u^2+v^2 & 2u & 2v \end{bmatrix} \, dudv = \int_{1\leqslant u^2+v^2\leqslant 2} \left(-2u^2 - 2v^2 - (u^2+v^2)\right) \, dudv \\ &= -3 \int_{1\leqslant u^2+v^2\leqslant 2} (u^2+v^2) \, dudv = -3 \int_0^{2\pi} \left(\int_1^{\sqrt{2}} \rho^2 \cdot \rho \, d\rho \right) d\theta = -6\pi \left[\frac{\rho^4}{4} \right]_{\rho=1}^{\rho=\sqrt{2}} = -\frac{9}{2}\pi. \quad \square\end{aligned}$$

A special but important case is when $\mathscr{M}$ is described as graph of a regular function, that is

$$\mathscr{M} := \{(x, y, \varphi(x, y)) \; : \; (x, y) \in D \subset \mathbb{R}^2\}.$$

In this case a natural parametrization is $\Phi(x, y) := (x, y, \varphi(x, y))$ and

$$\vec{n} = \frac{(1, 0, \partial_x \varphi) \times (0, 1, \partial_y \varphi)}{\|(1, 0, \partial_x \varphi) \times (0, 1, \partial_y \varphi)\|},$$

and being

$$(1, 0, \partial_x \varphi) \times (0, 1, \partial_y \varphi) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x \varphi \\ 0 & 1 & \partial_y \varphi \end{bmatrix} = (-\partial_x \varphi, -\partial_y \varphi, 1)$$

we obtain

$$\vec{n} = \frac{(-\partial_x \varphi, -\partial_y \varphi, 1)}{\sqrt{1 + \|\nabla \varphi\|^2}}.$$

The (5.4.4) take the form

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F} \cdot (-\partial_x \varphi, -\partial_y \varphi, 1) \, dxdy, \tag{5.4.5}$$

where, of course, $\vec{F} = \vec{F}(x, y, \varphi(x, y))$.

An important case of flux is the following. Suppose that the parametric surface $\mathscr{M}$ is the boundary of an open domain Ω, that is

$$\mathscr{M} = \partial\Omega.$$

At any point $\vec{x} \in \mathscr{M} = \partial\Omega$ we may expect that $\vec{n}(\vec{x})$ can point only in two directions: either *inward* or *outward*. Formalizing this intuitive idea is not easy, but, yet, this concept is quite natural. For example, it is possible to prove that

Proposition 5.4.3

Let $\Omega := \{(x, y, z) \in \mathbb{R}^2 \; : \; g(x, y, z) < 0\}$ with $g \in \mathscr{C}^1$ such that $\nabla g \neq \vec{0}$ on $\partial\Omega = \{g = 0\}$. Then

$$\vec{n}_e(x, y, z) = \frac{\nabla g(x, y, z)}{\|\nabla g(x, y, z)\|}, \; \forall (x, y, z) \in \partial\Omega.$$

PROOF. To show that ∇g is pointing outward, consider the straight line passing through point $\vec{x}$ with direction $\nabla g(\vec{x})$, that is

$$\vec{x} + t\nabla g(\vec{x}).$$

We claim that for $t > 0$ small, $\vec{x} + t\nabla g(\vec{x}) \notin \Omega$, thus this line leaves Ω at least for a while.

normext.pdf

To show this, let

$$\phi(t) := g(\vec{x} + t\nabla g(\vec{x})).$$

Clearly $\phi(0) = g(\vec{x}) = 0$ because $\vec{x} \in \partial\Omega$. Now,

$$\phi'(t) = \nabla g(\vec{x} + t\nabla g(\vec{x})) \cdot \nabla g(\vec{x}, \implies \phi'(0) = \nabla g(\vec{x}) \cdot \nabla g(\vec{x}) = \|\nabla g(\vec{x})\|^2 > 0.$$

Therefore, for some $\delta > 0$, we have $\phi' > 0$ on $[0, \delta]$, thus $\phi \nearrow$ on $[0, \delta]$ and since $\phi(0) = 0$ this means $\phi(t) = g(\vec{x} + t\nabla g(\vec{x})) > 0$. Thus, in particular, $\vec{x} + t\nabla g(\vec{x}) \notin \Omega$ for $t \in [0, \delta]$ as claimed. □

EXAMPLE 5.4.4. In the case of the ball $\Omega = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 < r^2\}$, $g(x, y, z) = x^2 + y^2 + z^2 - r^2$, hence $\nabla g = (2x, 2y, 2z) = \vec{0}$ iff $(x, y, z) = 0_3 \notin \partial\Omega$. Therefore

$$\vec{n}_e(x, y, z) = \frac{\nabla g}{\|\nabla g\|} = \frac{2(x, y, z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

It is easy to check directly that $\vec{n}_e$ is outward. □

5.5. Divergence theorem

Divergence theorem allows to transform the computation of an outward flux (which is, in general, not an easy task) into a volume integral (in general much easier).

Theorem 5.5.1: Gauss

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set equipped with outward normal. Let $\vec{F} = (f, g, h) : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^3$ be a $\mathscr{C}^1$ vector field on Ω and continuous on $\partial\Omega$. Then

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\Omega} \mathrm{div}\vec{F}. \tag{5.5.1}$$

where $\mathrm{div}\,\vec{F} = \partial_x f + \partial_y g + \partial_z h$ is called **divergence of** $\vec{F}$.

PROOF. The proof in the general case of Ω is hard and technical. We will do on a very simple case that however will show the basic idea. Let's consider Ω be a parallelepiped $\Omega =]a_1, b_1[\times]a_2, b_2[\times]a_3, b_3[$ and let's call $\mathscr{M}_1^{\pm}$, $\mathscr{M}_2^{\pm}$ and $\mathscr{M}_3^{\pm}$ the opposite sides.

flux1.jpg

Then

$$\langle\vec{F}\rangle_{\partial\Omega} = \int_{\mathscr{M}_1^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_2^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_2^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_3^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathscr{M}_3^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

Let's take the outward flux by $\mathscr{M}_1^{\pm}$ that can be described as

$$\mathscr{M}_1^+ = \{(b_1, y, z) \;:\; y \in [a_2, b_2],\ z \in [a_3, b_3]\}, \quad \mathscr{M}_1^- = \{(a_1, y, z) \;:\; y \in [a_2, b_2],\ z \in [a_3, b_3]\}.$$

It is clear that $\vec{n}_e = (1, 0, 0)$ on $\mathscr{M}_1^+$ while $\vec{n}_e = (-1, 0, 0)$ on $\mathscr{M}_1^-$. Therefore

$$\begin{aligned}
&\int_{\mathscr{M}_1^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 \\
&\quad= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f, g, h)\cdot(1,0,0)\, dydz + \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f, g, h)\cdot(-1,0,0)\, dydz \\
&\quad= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(b_1, y, z)\, dydz - \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(a_1, y, z)\, dydz \\
&\quad= \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} \left(f(b_1, y, z) - f(a_1, y, z)\right)\, dydz.
\end{aligned}$$

Now, by the fundamental formula of integral calculus

$$f(b_1, y, z) - f(a_1, y, z) = \int_{a_1}^{b_1} \partial_x f(x, y, z)\, dx,$$

hence

$$\int_{\mathscr{M}_1^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_1^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} \left(\int_{a_1}^{b_1} \partial_x f(x, y, z)\, dx\right) dydz = \int_\Omega \partial_x f.$$

Similarly

$$\int_{\mathscr{M}_2^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_2^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_y g, \quad \int_{\mathscr{M}_3^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathscr{M}_3^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_z h.$$

Summing up these formulas we obtain finally

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma = \int_\Omega \left(\partial_x f + \partial_y g + \partial_z h\right) = \int_\Omega \mathrm{div}\vec{F}. \qquad \square$$

Example 5.5.2. *Compute the outward flux by* $D = \left\{(x, y, z) \in \mathbb{R}^3 \;:\; \frac{\sqrt{x^2+(y/2)^2}-3}{4} + z < 1\right\}$ *of the vector field* $\vec{F}(x, y, z) = (x, y, z^2)$.

Sol. — By divergence thm

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \mathrm{div}\vec{F}\, dxdydz = \int_\Omega (1 + 1 + 2z)\, dxdydz = 2\int_\Omega dxdydz + 2\int_D z\, dxdydz.$$

The second integral vanishes being Ω invariant by symmetry $(x, y, z) \longmapsto (x, y, -z)$ while the integrand is odd. The first integral is instead the volume of Ω: by using adapted cylindrical coordinates,

$$\begin{cases} x = \rho\cos\theta, \\ \frac{y}{2} = \rho\sin\theta, \\ z = z, \end{cases} \iff \begin{cases} x = \rho\cos\theta, \\ y = 2\rho\sin\theta, \\ z = z. \end{cases}$$

we have

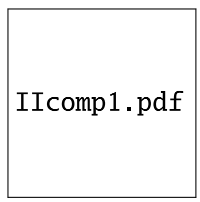
$$\Omega_{(\rho,\theta,z)} = \left\{(\rho,\theta,z) \in \mathbb{R}_+ \times [0,2\pi] \times \mathbb{R} \;:\; \frac{\rho-3}{4} + z^2 \leqslant 1, \iff \rho \leqslant 4(1-z^2)\right\}.$$

Notice that this last condition forces $z \in [-1,1]$ whence

$$\begin{aligned}\int_D dxdydz &= \int_{D_{(\rho,\theta,z)}} 2\rho\, d\rho d\theta dz = 2\int_{-1}^{1} dz\left(\int_0^{2\pi} d\theta\left(\int_0^{4(1-z^2)} \rho\, d\rho\right)\right) = 2\pi\int_{-1}^{1} 16(1-z^2)^2\, dz \\ &= 2\pi\int_{-1}^{1} 16(1-2z^2+z^4)\, dz = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right). \quad \square\end{aligned}$$

Example 5.5.3. *Let* $\Omega := \{(x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2 < z < 1 + \sqrt{1-(x^2+y^2)}\}$*. Compute the outward flux by* Ω *of the vector field* $\vec{F}(x,y,z) := (x,y,x^2+y^2)$*, computing also all the part of the flux relative to the several portions of* $\partial\Omega$*.*

Sol. — Ω is the region included between the paraboloid $z = x^2+y^2$ and the ball centered at $(0,0,1)$ with radius 2.



By divergence thm

$$\langle\vec{F}\rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma = \int_\Omega \mathrm{div}\vec{F}\, dxdydz = \int_\Omega (1+1+0)\, dxdydz = 2m_3(\Omega).$$

Let's compute $m_3(\Omega)$. In cylindrical coordinates

$$m_3(\Omega) = \int_\Omega dxdydz = \int_{\Omega_{cil}} \rho\, d\rho d\theta dz,$$

where

$$\Omega_{cil} = \left\{(\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R} \;:\; \rho^2 \leqslant z \leqslant 1+\sqrt{1-\rho^2}\right\}.$$

Of course one need that $1-\rho^2 \geqslant 0$, that is $0 \leqslant \rho \leqslant 1$ and also $\rho^2 \leqslant 1+\sqrt{1-\rho^2}$, always true being $\rho \leqslant 1$. Therefore

$$\begin{aligned}m_3(\Omega) &= \int_{0\leqslant\rho\leqslant1,\ 0\leqslant\theta\leqslant2\pi,\ \rho^2\leqslant z\leqslant1+\sqrt{1-\rho^2}} \rho\, d\rho d\theta dz \overset{FT}{=} \int_0^{2\pi}\left(\int_0^1\left(\int_{\rho^2}^{1+\sqrt{1-\rho^2}} \rho\, dz\right) d\rho\right) d\theta \\ &= 2\pi\int_0^1 \rho\left(1+\sqrt{1-\rho^2}-\rho^2\right) d\rho = 2\pi\left(\left[\frac{\rho^2}{2}\right]_0^1 + \left[-\frac{1}{3}(1-\rho^2)^{3/2}\right]_0^1 - \left[\frac{\rho^4}{4}\right]_0^1\right) \\ &= 2\pi\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right) = \frac{7}{6}\pi.\end{aligned}$$

Let's compute now the component of the flux outgoing by the paraboloid. This is described by the equation $z = x^2 + y^2$ and the part interior to Ω is $z > x^2 + y^2$, that is $g(x,y,z) := x^2 + y^2 - z < 0$. Therefore,

$$\vec{n}_e = \frac{\nabla g}{\|\nabla g\|} = \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2 + y^2)}}.$$

Hence, calling $\mathscr{P}$ the paraboloid, we have

$$\int_{\mathscr{P}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\mathscr{P}} (x, y, x^2 + y^2) \cdot \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2 = \int_{\mathscr{P}} \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2.$$

Now, $\mathscr{P}$ is the graph of $(x, y) \longmapsto x^2 + y^2$ on the domain $D = x^2 + y^2 \leqslant 1$: therefore

$$\begin{aligned} \int_{\mathscr{P}} \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \, d\sigma_2 &= \int_D \frac{x^2 + y^2}{\sqrt{1 + 4(x^2 + y^2)}} \sqrt{1 + 4(x^2 + y^2)} \, dxdy = \int_D (x^2 + y^2) \, dxdy \\ &= \int_0^1 \left(\int_0^{2\pi} \rho^3 \, d\theta \right) d\rho = \frac{\pi}{2}. \end{aligned}$$

The remaining component of the flux (that one relative to the half sphere) can be now deduce by difference by the total outward flux. □

5.5.1. Gauss theorem on central fields. A beautiful application of the divergence theorem is the beautiful Gauss theorem on outward flux of a central field. This result is very important for gravitation and electric forces.

Theorem 5.5.4: Gauss

Let

$$\vec{F}(x, y, z) = F_0 \frac{(x, y, z)}{\|(x, y, z)\|^3}, \quad (x, y, z) \in \mathbb{R}^3 \backslash \{0_3\}.$$

Then, for any open set $\Omega \subset \mathbb{R}^3$,

$$\langle \vec{F} \rangle_{\partial\Omega} = \begin{cases} 0, & \text{if } 0_3 \in \Omega^c \backslash \partial\Omega, \\ \\ 4\pi F_0, & \text{if } 0_3 \in \Omega. \end{cases}$$

Proof. Assume first that $0_3 \in \Omega^c \backslash \partial\Omega$. Then $\vec{F} \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\partial\Omega)$. By divergence thm

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\Omega} \mathrm{div}\vec{F} \, dxdydz.$$

Now: the key point is that $\mathrm{div}\vec{F} = 0$. Indeed,

$$\begin{aligned} \mathrm{div}\vec{F} &= F_0 \left(\partial_x \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + \partial_y \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \partial_z \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right) \\ &= F_0 \left(\frac{(x^2 + y^2 + z^2) - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(x^2 + y^2 + z^2) - 3z^2}{(x^2 + y^2 + z^2)^{5/2}} \right) = 0. \end{aligned}$$

Therefore we deduce $\langle \vec{F} \rangle_{\partial\Omega} = 0$.

Let's pass to the other case: $0_3 \in \Omega$. In this case we cannot say that $\vec{F} \in \mathscr{C}^1(\Omega)$ because Ω contains 0_3 which is the singularity of $\vec{F}$, so we cannot apply the divergence thm on Ω. However we can take out a suitably small ball $B(0_3, \rho] \subset \Omega$ (it exists because Ω is open and $0_3 \in \Omega$) and define

$$\widetilde{\Omega} := \Omega \backslash B(0, \rho].$$

Gauss.jpg

It is clear that $\widetilde{\Omega}$ is open and $\partial\widetilde{\Omega} = \partial\Omega \cup \partial B(0, \rho]$ and now $\vec{F} \in \mathscr{C}^1(\widetilde{\Omega}) \cap \mathscr{C}(\partial\widetilde{\Omega})$. By the divergence thm then

$$0 = \int_{\widetilde{\Omega}} \mathrm{div}\vec{F} = \int_{\partial\widetilde{\Omega}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 - \int_{\partial B(0,\rho]} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

The agreement here is that in the second integral $\vec{n}_e$ stand for the outward normal unit vector by $B(0, \rho]$ (therefore the outward normal by the ball is inward for $\widetilde{\Omega}$. Now, because

$$\int_{\partial B(0,\rho]} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\partial B(0,\rho]} F_0 \frac{(x, y, z)}{\|(x, y, z)\|^3} \cdot \frac{(x, y, z)}{\|(x, y, z)\|} \, d\sigma_2 = F_0 \int_{\partial B(0,\rho]} \frac{1}{\|(x, y, z)\|^2} \, d\sigma_2 = F_0 \frac{1}{\rho^2} 4\pi\rho^2 = 4\pi F_0,$$

we conclude that $\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = 4\pi F_0$. □

Let's consider for instance the case of the electric field generated by a point charge q_0 at point (x_0, y_0, z_0), that is

$$\vec{E}(x, y, z) = \frac{q_0}{\varepsilon} \frac{(x - x_0, y - y_0, z - z_0)}{\|(x - x_0, y - y_0, z - z_0)\|}$$

where ε is the *permittivity*. By the Gauss thm

$$\langle \vec{E} \rangle_{\partial\Omega} = \begin{cases} 4\pi \frac{q_0}{\varepsilon}, & \text{if } (x_0, y_0, z_0) \in \Omega, \\ 0, & \text{if } (x_0, y_0, z_0) \in \Omega^c \backslash \partial\Omega. \end{cases}$$

Let $\varrho = \varrho(x, y, z)$ be now a charge density. We'll have

$$\langle \vec{E} \rangle_{\partial\Omega} = \frac{4\pi}{\varepsilon} \int_\Omega \varrho.$$

On the other hand, by divergence thm,

$$\langle \vec{E} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{E} \cdot \vec{n}_e \, d\sigma = \int_\Omega \mathrm{div}\vec{E},$$

whence

$$\int_\Omega \mathrm{div}\vec{E} = \int_\Omega \frac{4\pi}{\varepsilon} \varrho, \; \forall\Omega, \iff \mathrm{div}\vec{E} = \frac{4\pi}{\varepsilon} q.$$

This is the first of Maxwell equations, the fundamental equations of the electro-magnetic field.

5.5.2. Gradient theorem. By the divergence thm we have the

Corollary 5.5.5

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set equipped with outward normal and let $f : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}$ be a function $\mathscr{C}^1$ on Ω and continuous on $\partial\Omega$. Then

$$\int_\Omega \nabla f = \int_{\partial\Omega} f \vec{n}_e \, d\sigma_2.$$

Proof. Let's check the first component of the stated identity, that is

$$\int_\Omega \partial_x f = \int_{\partial\Omega} f (\vec{n}_e)_x$$

where $(\vec{n}_e)_x$ is the first component of $\vec{n}_e$. To this aim introduce the vector field $\vec{F} := (f, 0, 0)$. Then,

$$\mathrm{div}\vec{F} = \partial_x f + \partial_y 0 + \partial_z 0 = \partial_x f,$$

whence

$$\int_\Omega \partial_x f = \int_\Omega \mathrm{div}\vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e = \int_{\partial\Omega} (f, 0, 0) \cdot \vec{n}_e = \int_{\partial\Omega} f (\vec{n}_e)_x. \quad \square$$

A nice application of this Corollary is the well known

Corollary 5.5.6: Archimedean principle

A body immersed in an incompressible homogeneous fluid experiences a buoyant force equal to the weight of the displaced fluid.

Proof. Assume that the fluid has a constant density per unit of volume ϱ and let $p = p(x, y, z)$ the pressure (weight force per unit of surface) exercised by the fluid at point (x, y, z). For simplicity we may assume that the z axis corresponds to the vertical axis and $z = 0$ correspond to the surface of the fluid (assumed to be flat). Therefore, if p_0 (constant) is the atmospheric pressure on the surface of the fluid,

$$p(x, y, z) = -\varrho g z + p_0.$$

Now, assume that the body is represented by an open and bounded set $\Omega \subset \{z \leqslant 0\}$. The resultant of the pressure forces on the body is therefore

$$-\int_{\partial\Omega} p \vec{n}_e \, d\sigma_2.$$

The $-$ gives account of the fact that are counted positively pressures inward directed to Ω. Now, by the gradient thm

$$-\int_{\partial\Omega} p \vec{n}_e \, d\sigma_2 = -\int_\Omega \nabla p \, dxdydz = -\int_\Omega (0, 0, -\varrho g) \, dxdydz = \varrho g m_3(\Omega) \vec{k},$$

that is the resultant of the pressure forces is upward oriented with intensity $\varrho g m_3(\Omega)$ which is nothing but the weight of the body. $\square$

5.6. Green formula

The divergence theorem hods also in the case of a plane domain $\Omega \subset \mathbb{R}^2$. In this case $\partial\Omega$ is a $1-$dim. surface, that is a curve that can be parametrized by $\gamma = \gamma(t) : [a,b] \longrightarrow \mathbb{R}^2$. Proceeding similarly to the surface integrals, we define

$$\int_{\partial\Omega} f \, d\sigma_1 := \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

With this agreement,

Theorem 5.6.1

Let $\Omega \subset \mathbb{R}^2$ be a stokian open and bounded set and let $\vec{F} = (f,g) : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^2$ be a $\mathscr{C}^1$ vector field on Ω continuous on $\partial\Omega$. Then

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_1 = \int_{\Omega} \mathrm{div}\vec{F}. \tag{5.6.1}$$

where $\mathrm{div}\vec{F} = \partial_x f + \partial_y g$ is called **divergence of** $\vec{F}$.

Now, assume that $\partial\Omega \equiv \gamma([a,b])$ where $\gamma : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ be a regular circuit (that is $\partial\Omega$ be the trace of the circuit γ).

Green1.pdf

We notice that if $\gamma(t) = (x(t), y(t))$, then $\gamma'(t) = (x'(t), y'(t))$ is tangent to γ hence the vectors

$$(-y'(t), x'(t)), \quad (y'(t), -x'(t))$$

are perpendicular to γ. Of course, being opposite, one will be outward and the other inward to Ω. The question is, of course: which one is the outward? The intuition suggests that *if γ is counter-clock wise oriented, then $(y'(t), -x'(t))$ will be outward by Ω, otherwise it will be $(-y'(t), x'(t))$ the outward vector*. Assuming the counter-clock orientation and that $\gamma' \neq 0$ always,

$$\vec{n}_e(x(t), y(t)) = \frac{(y'(t), -x'(t))}{\|(y'(t), -x'(t))\|} = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}} = \frac{(y'(t), -x'(t))}{\|\gamma'(t)\|}.$$

In this way

$$\begin{aligned} \int_{\partial\Omega} (f,g) \cdot \vec{n}_e \, d\sigma_1 &= \int_a^b (f,g) \cdot \frac{(y'(t), -x'(t))}{\|\gamma'(t)\|} \|\gamma'(t)\| \, dt = \int_a^b (f,g) \cdot (y', -x') = \\ &= \int_a^b (-g, f) \cdot (x', y') = \oint_\gamma (-g, f). \end{aligned}$$

By this remark we derive the

Theorem 5.6.2: Green

Let $\vec{F} = (f, g)$ be a vector field $\mathscr{C}^1(\Omega)$ with $\Omega \subset \mathbb{R}^2$ open and bounded and such that $\partial\Omega \equiv \gamma$ be a circuit. Then

$$\oint_\gamma \vec{F} = \int_\Omega (\partial_x g - \partial_y f) \tag{5.6.2}$$

Proof. By the preliminaries we have that

$$\oint_\gamma \vec{F} = \int_a^b (f, g) \cdot (x', y') = \int_{\partial\Omega} (g, -f) \cdot \vec{n}_e \, d\sigma_1 = \int_\Omega \operatorname{div}(g, -f) = \int_\Omega (\partial_x g - \partial_y f). \quad \square$$

Example 5.6.3. *Compute the counter-clock circulation of* $\vec{F} = (y^3, -x^3)$ *on* $\gamma = \partial B(0, 2]$.

Sol. — By Green's formula

$$\begin{aligned}\oint_\gamma \vec{F} &= \int_{x^2+y^2\leqslant 4} \left(\partial_x(-x^3) - \partial_y(y^3)\right) \, dxdy = -3 \int_{x^2+y^2\leqslant 4} (x^2 + y^2) \, dxdy = -3 \int_0^{2\pi} \left(\int_0^2 \rho^2 \, d\rho \right) \, d\theta \\ &= -6\pi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=2} = -16. \quad \square\end{aligned}$$

As particular case of the Green formula we obtain the

Corollary 5.6.4: Area formula

Let $\Omega \subset \mathbb{R}^2$ be open and bounded such that $\partial\Omega \equiv \gamma$, with $\gamma \in \mathscr{C}^1$. Then

$$\lambda_2(\Omega) = \oint_\gamma (0, x) \equiv \oint_\gamma x \, dy = -\oint_\gamma y \, dx. \tag{5.6.3}$$

Proof. By the Green formula

$$\oint_\gamma (0, x) = \int_\Omega \operatorname{div}(x, -0) = \int_\Omega 1 = \lambda_2(\Omega). \quad \square$$

Example 5.6.5. *Compute the area* Ω *delimitated by the cycloid* $(a(t - \sin t), a(1 - \cos t))$, $t \in [0, 2\pi]$ *and the* $x-$*axis.*

Sol. — Let's apply the area formula $\lambda_2(\Omega) = -\oint_{\partial\Omega} y \, dx$. The part of this integral relative to the part of $\partial\Omega$ on the axis vanishes because $y \equiv 0$. Considering also the orientation of the cycloid,

$$\lambda_2(\Omega) = \int_0^{2\pi} a(1 - \cos t) \cdot a(1 - \cos t) \, dt = -a^2 \int_0^{2\pi} (1 - \cos t)^2 \, dt.$$

Now, $\int_0^{2\pi}(1-\cos t)^2\,dt = 2\pi - 2\int_0^{2\pi}\cos t\,dt + \int_0^{2\pi}(\cos t)^2\,dt = 2\pi + \int_0^{2\pi}(\cos t)^2\,dt$ hence,

$$\begin{aligned}\int_0^{2\pi}(\cos t)^2\,dt &= \int_0^{2\pi}\cos t(\sin t)'\,dt = [\cos t\sin t]_{t=0}^{t=2\pi} - \int_0^{2\pi}(-\sin t)\sin t\,dt = \int_0^{2\pi}(1-(\cos t)^2)\,dt\\ &= 2\pi - \int_0^{2\pi}(\cos t)^2\,dt,\end{aligned}$$

so, finally, $\int_0^{2\pi}(\cos t)^2\,dt = \pi$. We obtain $\lambda_2(\Omega) = 3a^2\pi$. □

5.7. Stokes Formula

Green's formula transform a plane circulation along $\gamma \subset \mathbb{R}^2$ into a plane integral on a domain D such that its boundary coincides with the circuit γ. *Stokes' formula* is the extension of Green's formula to the case of a circulation along $\gamma \subset \mathbb{R}^3$. Let's start with the

Definition 5.7.1

We say that a parametric surface $\mathscr{M} := \Phi(D) \subset \mathbb{R}^3$ has an **edge** if $\partial D \equiv \gamma$ circuit. The image of γ through Φ is called edge of $\mathscr{M}$ and it is denoted by $\partial\mathscr{M}$ [a]. We say that $\partial\mathscr{M}$ is counter clock wise oriented iff γ is counter-clock wise oriented w.r.t. D.

[a] Warning: here $\partial\mathscr{M}$ is not the boundary of $\mathscr{M}$.

Stokes1.pdf Surf2.pdf Stokes2.pdf

We have the

Theorem 5.7.2: Stokes'

Let $\mathscr{M}$ be a parametric surface with edge counter-clock wise oriented and let $\vec{F} : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a $\mathscr{C}^1$ vector field on $D \supset \mathscr{M}$. Then

$$\oint_{\partial\mathscr{M}}\vec{F} = \int_{\mathscr{M}}(\nabla\times\vec{F})\cdot\vec{n}\,d\sigma_2, \tag{5.7.1}$$

where

$$\nabla\times\vec{F} := \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial_x & \partial_y & \partial_z\\ f & g & h\end{bmatrix} = \left(\partial_y h - \partial_z g, \partial_z f - \partial_x h, \partial_x g - \partial_y f\right).$$

In other words: the circulation of $\vec{F}$ is the flux of $\nabla\times\vec{F}$ (called **curl** of $\vec{F}$) through $\mathscr{M}$.

Proof. Let's limit to a special (but important) case: $\mathscr{M}$ be the graph of a regular function,

$$\mathscr{M} = \{(x,y,\varphi(x,y)) \;:\; (x,y) \in D \subset \mathbb{R}^2\},\ \text{with } \partial D \equiv \gamma.$$

Assume that $\gamma = \gamma(t) = (x(t), y(t))$ be counter-clock wise oriented. Then

$$\oint_{\partial\mathscr{M}}\vec{F} = \int_a^b (f,g,h)\cdot(x,y,\varphi(x,y))'.$$

Noticed that

$$(x, y, \varphi(x, y))' = (x', y', \partial_x\varphi x' + \partial_y\varphi y'),$$

we have

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_a^b \left(fx' + gy' + h(\partial_x\varphi x' + \partial_y\varphi y')\right) = \int_a^b (f + h\partial_x\varphi, g + h\partial_y\varphi) \cdot (x', y') = \oint_{\partial D} (f + h\partial_x\varphi, g + h\partial_y\varphi).$$

By Green formula then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D \left(\partial_x \left(g + h\partial_y\varphi\right) - \partial_y \left(f + h\partial_x\varphi\right)\right).$$

Now, recall that every of f, g, h is evaluated at $(x, y, \varphi(x, y))$. Therefore

$$\partial_x \left(g + h\partial_y\varphi\right) = \partial_x g + \partial_z g\partial_x\varphi + (\partial_x h + \partial_z h\partial_x\varphi)\partial_y\varphi + h\partial_{xy}\varphi,$$

$$\partial_y \left(f + h\partial_x\varphi\right) = \partial_y f + \partial_z f\partial_y\varphi + (\partial_y h + \partial_z h\partial_y\varphi)\partial_x\varphi + h\partial_{xy}\varphi,$$

and by taking their difference we obtain the quantity

$$(\partial_x g - \partial_y f) + \partial_x\varphi(\partial_z g - \partial_y h) + \partial_y\varphi(\partial_x h - \partial_z f) = \nabla \times (f, g, h) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1),$$

hence

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D (\nabla \times (f, g, h)) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1)\, dxdy \overset{(5.4.5)}{=} \int_{\mathscr{M}} (\nabla \times (f, g, h)) \cdot \vec{n}\, d\sigma_2. \quad \square$$

Example 5.7.3. *Compute the circulation of* $\vec{F}(x, y, z) := (x^2 z, xy^2, z^2)$ *along* $\gamma = \{x + y + z = 1,\ x^2 + y^2 = 9\}$.

Sol. — Clearly γ is an ellipse that we may think as edge of

$$\mathscr{M} = \Phi(D),\ \ \Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ \ \Phi(x, y) = (x, y, 1 - (x + y)),\ \ D := \{(x, y) : x^2 + y^2 \leqslant 9\}.$$

It is easy to take the normal unit vector to $\mathscr{M}$ in such a way that $\gamma = \partial\mathscr{M}$ be counter-clock wise oriented: $\vec{n} = \frac{(1,1,1)}{\sqrt{3}}$. Moreover, being $\mathscr{M}$ graph of $f(x, y) := 1 - (x + y)$ $(x, y) \in D$, we have

$$d\sigma_2 = \sqrt{1 + \|\nabla f(x, y)\|^2}\, dxdy = \sqrt{1 + \|(-1, -1)\|^2}\, dxdy = \sqrt{3}\, dxdy.$$

Therefore being $\nabla \times \vec{F} = (0, x^2, y^2)$, by the curl theorem

$$\begin{aligned}\oint_\gamma \vec{F} &= \int_{\mathscr{M}} \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = \int_{x^2+y^2\leqslant 9} (0, x^2, y^2) \cdot \frac{(1, 1, 1)}{\|(1, 1, 1)\|} \sqrt{3}\, dxdy = \int_{x^2+y^2\leqslant 9} (x^2 + y^2)\, dxdy \\ &= \int_0^{2\pi} d\theta \int_0^3 \rho^3\, d\rho = 2\pi \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=3} = \pi\frac{3^4}{2}. \quad \square\end{aligned}$$

5.8. Exercices

Exercise 5.8.1. For each of the following surfaces, determine a parametrization and check if these are parametric surfaces:

i) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ :\ x + y + z = 1,\ x^2 + y^2 \leqslant 1\}$.
ii) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ :\ x^2 + y^2 - z^2 = 1\}$.
iii) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \ :\ z^2 = xy + 1\}$

Exercise 5.8.2. Compute the area of each of the following surfaces:

- $\mathscr{M} := \left\{(x; y; z) \in \mathbb{R}^3 \ :\ x \geqslant 0,\ z \geqslant 0,\ x + z = 2,\ x^2 + y^2 - 2x \leqslant 0\right\}$.

- $\mathscr{M} := \left\{(x, y, z)\ : z \geqslant 0,\ x^2 + y^2 + z^2 = r^2, \left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}\right\}$ (Viviani's vault).
- $\mathscr{M}$ rotation surface around the $x-$axis of the graph of the function $f(x) = \alpha x^2$, $x \in [0, a]$, $a, \alpha > 0$.
- $\mathscr{M}$ rotation surface around the $z-$axis of the graph of the function $f(y) = 2 - \cosh y$, $y \in [0, a]$, a such that $f \geqslant 0$.

Exercise 5.8.3. Let $\mathscr{M}$ be the rotation surface around the $z-$axis of the curve $y = ze^{-z}$, $z \in [0, h]$ $(h > 0)$. Compute the outward flux of $\vec{F}(x, y, z) := (x + y^2, y + x^2, z)$ through $\mathscr{M}$.

Exercise 5.8.4. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3\ :\ 1 \leqslant z \leqslant 2,\ \sqrt{x^2 + y^2} \leqslant z\}$. Draw Ω and compute the outward flux of $\vec{F}(x, y, z) := (x^3, 0, z^2)$ by Ω.

Exercise 5.8.5. Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3\ :\ (x - z)^2 + (y + z)^2 \leqslant z^2,\ 0 \leqslant z \leqslant b\right\}$ $(b > 0)$. Compute the outward flux of $\vec{F}(x, y, z) = (xy, x - y, yz)$ by Ω.

Exercise 5.8.6. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant 1\}$. Draw Ω, compute its volume and the area of $\partial\Omega$. Hence, compute the outward flux of $\vec{F}(x, y, z) := (x, y, z)$ by Ω, computing the components of the flux by each part of $\partial\Omega$.

Exercise 5.8.7. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1,\ a \leqslant z \leqslant 1\}$ with $0 < a < 1$. Draw Ω, compute its volume and its barycenter. Compute the outward flux of $\vec{F}(x, y, z) := (x^2, -y^2, z)$ by Ω and each of its components on the several parts of $\partial\Omega$.

Exercise 5.8.8. Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leqslant e^y, 0 \leqslant y \leqslant a\}$ with $a > 0$. Draw Ω and compute outward flux and relative components of $\vec{F}(x, y, z) := (-z, (x^2 + z^2)y, x)$ by Ω.

Exercise 5.8.9. Let $\mathscr{M}$ the rotation surface around the $z-$axis of the curve $z = 4 - \cosh(y - 1)$, $y \in [1, 2]$. Calculate its area. Let now

$$\vec{F}(x, y, z) := \frac{(x, y, z)}{\|(x, y, z)\|^3} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}},\ \ (x, y, z) \in \mathbb{R}^3 \backslash \{0_3\}.$$

Compute the flux of $\vec{F}$ through $\mathscr{M}$ (hint: introduce an auxiliary disk $\left\{(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 = r^2,\ z = a\right\}$ and apply suitably the divergence theorem. . .).

Exercise 5.8.10. Let $\mathscr{M}$ the rotation surface around the $z-$axis of the curve $y = z^2$ $z \in [0, h]$ $(h > 0)$. Compute the area of $\mathscr{M}$ and the flux of $\vec{F}(x, y, z) := (xz, yz, -z)$ through $\mathscr{M}$.

Exercise 5.8.11. Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3\ :\ z^2 + 6 \leqslant x^2 + y^2 \leqslant 5z\right\}$. Is Ω is a rotation solid? Draw Ω, compute its volume and the outward flux of $\vec{F}(x, y, z) := (x^2, y, z^2)$ by Ω.

Exercise 5.8.12. Let $D := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ \sqrt{1 + x^2} \leqslant y \leqslant 2\}$ and let Ω the solid obtained by rotating D around the $y-$axis. Compute the area of $\partial\Omega$ and the outward flux by Ω of $\vec{F}(x, y, z) := (x^2, y, z^2)$.

Exercise 5.8.13. Let $\Omega := \{(x, y, z) \in [0, +\infty[^3 :\ 1 \leqslant x^2 + y^2 \leqslant 4 - z^2\}$. Compute the outward flux by Ω of $\vec{F}(x, y, z) := (x, y, z^2)$. Determine in particular the component of this flux on $\partial\Omega \cap \{x^2 + y^2 + z^2 = 4\}$.

Exercise 5.8.14. Compute the outward flux by $\Omega = \{(x, y, z) \in \mathbb{R}^3\ :\ \max\{|x|, |y|, |z|\} \leqslant 1\}$ of the vector field $\vec{F}(x, y, z) = \frac{(x,y,z)}{|(x,y,z)|^3}$.

Exercise 5.8.15. Let $\vec{F}(x, y, z) := (y, x, z)$. Show that g is conservative and find all its potentials on $\mathbb{R}^3$. Now, let

$$\vec{H}(x, y, z) = u\left(x^2 + y^2\right)\vec{F}(x, y, z),\ (x, y, z) \in \mathbb{R}^3 \backslash \{(0, 0, z)\ :\ z \in \mathbb{R}\},$$

where $u \in \mathscr{C}^1(]0, +\infty[)$.

i) Compute $\oint_\gamma \vec{H}$ where $\gamma(t) := (r\cos t, r\sin t, k)$, $t \in [0, 2\pi]$, $r > 0$ and $k \in \mathbb{R}$ fixed parameters.
ii) Find all the possible u such that $\vec{H}$ be irrotational.
iii) Find all the possible u such that $\vec{H}$ be conservative.

EXERCISE 5.8.16. By applying the Green formula compute

1) $\displaystyle\oint_\gamma (xy, 3x + 2y),\ \ \gamma \equiv \partial[-1, 1]^2.$

2) $\displaystyle\oint_\gamma (\cos x + 6y^2, 3x - e^{-y^2}),\ \ \gamma \equiv \partial B(0, 1].$

3) $\displaystyle\oint_\gamma \left(x^2 + y, xy\right),\ \ \gamma(t) := (1 + \cos t, \sin t),\ t \in [0, 2\pi].$

4) $\displaystyle\oint_\gamma (x^3 - y^3, x^3 + y^3),\ \ \gamma = \partial\left(B(0, r] \cap [0, +\infty[^2\right).$

EXERCISE 5.8.17. Compute the area delimitated by the following curves:

1) $t(\cos t, \sin t),\ t \in [0, 2\pi]$. 2) $(\sin t + \sin^2 t, -\cos t - \sin t\cos t),\ t \in [0, 2\pi]$. 3) $(\cos^2 t, \sin^2 t),\ t \in [0, 2\pi]$.

4) $(\cos^3 t, \sin^3 t),\ t \in [0, 2\pi]$. 5) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a, b > 0)$.

EXERCISE 5.8.18. By applying the curl theorem compute the $\oint_\gamma \vec{F}$ in the following cases:

1) $\gamma \equiv \begin{cases} x^2 + y^2 = 1, \\ x + z = 1, \end{cases} \quad \vec{F} = (y - z, z - x, x - y).$

2) $\gamma \equiv \begin{cases} x + y + z = 1, \\ x^2 + y^2 = 9, \end{cases} \quad \vec{F} = (x^2 z, xy^2, z^2)$

3) $\gamma \equiv \begin{cases} x^2 + z^2 = 1, \\ y = x^2, \end{cases} \quad \vec{F} = (x^3 + z^3, y^2, x^3 + z^3).$

4) $\gamma \equiv \begin{cases} x^2 + y^2 = 1, \\ z = xy, \end{cases} \quad \vec{F} = (y, z, x)$

5) $\gamma \equiv \begin{cases} x^2 + y^2 + z^2 = 1, \\ z = \sqrt{x^2 + y^2}. \end{cases} \quad \vec{F} = (y^2 + z^2, z^2 + x^2, x^2 + y^2).$

CHAPTER 6

Basic Differential Equations

Ordinary Differential Equations (ODEs) is a wide branch of Mathematical Analysis which is relevant in many applications to Physics, Engineering, Biology, Economy, etc. An ODE is

- an *equation*, that is an identity with an unknown,
- the unknown is a *function* of one real variable, say $y = y(t)$,
- the unknown appears in the equation together with some (but not necessarily all) of its *derivatives* $y'(t), y''(t), \ldots$ up to a certain maximum order, called *order of the equation*.

This explains the D and E of ODE. The O is to distinguish these equations from similar equations but with unknown function depending on several variables (this type of equation is called *Partial Differential Equations*, PDEs). PDEs accomplish a similar scope as ODEs, that is, describe certain real phenomena, but, of course, they are much more complicated and out of scope here.

As for every equation, we may expect that the principal problem connected to an ODE is to *determine its solutions*. In certain simple cases this task can be achieved analytically, but, as happens for algebraic equations, in general it is impossible to solve explicitly any equation. This leads to the development of suitable methods to study solutions of an ODE: *qualitative methods* and *numerical methods*.

The aim of this chapter is to introduce to the simplest types of ODEs: first- and second-order linear equations and first-order separable variable equations. These are explicitly solvable equations that cover some important applications. To understand the importance of these equations, we will accompany the theory with several applied examples. In the next Chapter we will focus on general remarks and tools for studying ODEs that cannot be explicitly solved.

Chapter requirements: a good understanding of the concept of derivative and computing primitives.

6.1. How Differential Equations arise

In this Section, we motivate why Differential Equations are so important in applications and how they arise starting from applied problems. The reader should focus on the *modeling process* that leads to writing an ODE starting from a modeling problem.

Example 6.1.1: Demographic models

A certain population can be described through the evolution of its size as a function of time, namely $P = P(t)$ represents the number of individuals of which the population is made at time t. The variation of the population over a period of time $[t, t + dt]$ is $P(t + dt) - P(t)$, while the **growth rate** over the same period of time is

$$r = \frac{P(t + dt) - P(t)}{dt \cdot P(t)}.$$

The **instantaneous growth rate** is defined by letting $dt \longrightarrow 0$. Assuming $P = P(t)$ a nice function of t (namely, differentiable), we have

$$r = \frac{P'(t)}{P(t)}.$$

The instantaneous growth rate gives a precise information on the behaviour of the population. For instance: a *flat rate*, that is $r \equiv 0$, means that

$$\frac{P'(t)}{P(t)} \equiv 0, \implies P'(t) \equiv 0, \implies P(t) \equiv P(0),$$

that is the population remains constant in time. Since here we assume $P > 0$, a positive/negative r means a growing/decreasing population. The instantaneous growth rate may depend on several factors. The two major of these are:

- time t, because external condition may be variable in time and what is the scenario of today could not be valid tomorrow, implying different growths (for example, thinking to a human population, this is the case of climatic effect, wars, pandemics, etc);
- population $P(t)$, because the size of the population may affect the growth (for example, for a larger population there will be less food).

In other words, a natural shape for r is $r = r(t, P(t))$. This leads to an equation

$$\boxed{\frac{P'(t)}{P(t)} = r(t, P(t)),} \tag{6.1.1}$$

which is a differential equation in the unknown $P = P(t)$.

The typical problem considered in this modeling is the **prediction problem**: *given the (known) population at a certain initial time (conventionally taken as $t = 0$), make a prediction / forecast on the population $P(t)$ in the future (at times $t > 0$).* We can formalize this in the following way: solve the following problem.

$$\begin{cases} \frac{P'(t)}{P(t)} = r(t, P(t)), \\ \\ P(0) = p_0, \end{cases}$$

where p_0 is known. This problem is called **Cauchy problem** or **initial value problem**. We illustrate this in two important cases here below.

6.1.1. Malthus model. The simplest possible assumption on the instantaneous growth rate is to take it constant, namely, $r(t, P(t)) \equiv r_0$. Of course, this is an extreme simplification, yet a reasonable assumption in certain conditions. For example, the growth rate of a human population is slowly variable in time, hence for a short period taking r constant is not an improper assumption. Thus we consider the problem

$$\begin{cases} \frac{P'(t)}{P(t)} = r_0, \\ \\ P(0) = p_0, \end{cases}$$

Let us focus on the equation. We may notice that

$$r_0 = \frac{P'(t)}{P(t)} = (\log|P(t)|)',$$

thus

$$\log|P(t)| = \int r_0\,dt + c = r_0 t + c,$$

for some constant c. The value of the constant can be determined imposing the initial condition:

$$\log|P(0)| = c,$$

thus

$$\log|P(t)| = r_0 t + \log|P(0)|, \iff |P(t)| = e^{r_0 t + \log|P(0)|} = |P(0)|e^{r_0 t}.$$

Here, P represents a population size, thus $P(t) > 0$. We conclude that

$$P(t) = P(0)e^{r_0 t}.$$

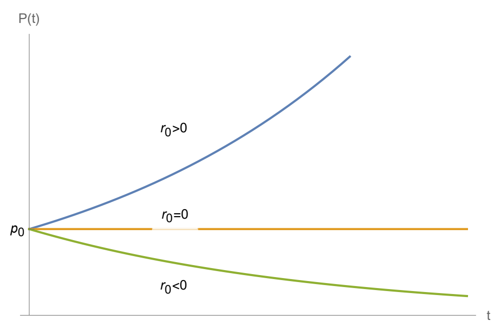


FIGURE 1. Malthus model possible scenarios

Notice that this leads to three possible behaviours:

- if $r_0 > 0$, the population grows exponentially, and $P(t) \longrightarrow +\infty$ when $t \longrightarrow +\infty$;
- if $r_0 = 0$, the population remains constant $P(t) \equiv p_0$;
- if $r_0 < 0$, the population decays exponentially to the extinction, $P(t) \longrightarrow 0$ when $t \longrightarrow +\infty$.

6.1.2. Logistic model. Malthus' model is too simplistic. Thinking to a real population, if the growth rate r_0 is positive, the population grows without any bound. This might be conflicting with environment physical limits (for example, the physical space for the population is normally bounded, or a large population might not have enough food to survive). For this reason, a more realistic growth rate should depend on the size of the population in such a way that the larger is the population, the lower is the growth rate. The simplest way to express this dependence is through a linear function

$$r = a(m - P(t)).$$

What is the interpretation of constants a and m? First, we may notice that, assuming $a > 0$, $r > 0$ iff $P < m$. Thus, m represents a *threshold* beyond which the growth is negative. In other words, m may represent the maximal population size the environment can tolerate. Since $m - P$ is a size, a represent a growth rate per unit of population.

Thus, Malthus' equation (6.1.1) becomes

$$\begin{cases} \frac{P'(t)}{P(t)} = a(m - P(t)), \\ \\ P(0) = p_0. \end{cases}$$

Let us focus again on the equation. Here the solution is more involved, yet it follows the same idea of Malthus' model. For ease of notations, we write P for $P(t)$ and P' for $P'(t)$. Notice that, until $P \neq m$, the equation can be written in the form

$$\frac{P'}{P(m-P)} = a. \tag{6.1.2}$$

As in the previous case, the idea is to recognize that the l.h.s. is a derivative. To show this, notice that

$$\frac{1}{P(m-P)} = \frac{1}{m}\left(\frac{1}{m-P} + \frac{1}{P}\right),$$

thus

$$\frac{P'}{P(m-P)} = \frac{1}{m}\left(\frac{P'}{m-P} + \frac{P'}{P}\right) = \frac{1}{m}\left(-\log|m-P| + \log|P|\right)' = \frac{1}{m}\left(\log\frac{|P|}{|m-P|}\right)'$$

Returning on equation (6.1.2), we have

$$\left(\log\frac{|P|}{|m-P|}\right)' = am, \implies \log\frac{|P|}{|m-P|} = amt + c,$$

for some constant c. At this stage, we have still an equation in the unknown P, but now this is no more a differential equation. Rather, we have to solve an algebraic equation to extract P. We may say that we found P in *implicit form*.

As for Malthus' equation, the value of constant c can be determined by imposing the passage condition. Here we may notice that, apart for $P(0) = m$, the solution can be explicitly found. Assume for instance that $0 < P(0) < m$. Until $0 < P(t) < m$ (by continuity, this is certainly true for some time $t \in [0, T[$) we have

$$\log\frac{P}{m-P} = amt + c, \longrightarrow c = \log\frac{P(0)}{m-P(0)},$$

and

$$\frac{P}{m-P} = \frac{p_0}{m-p_0}e^{amt}, \implies P = \frac{mp_0}{m-p_0}\frac{e^{amt}}{1 + \frac{p_0}{m-p_0}e^{amt}}.$$

It is not difficult to check that that $p_0 < p(t) < m$ for all times $t \in]0, +\infty[$, $p(t)$ is increasing with t and

$$p(t) \longrightarrow m, \text{ when } t \longrightarrow +\infty.$$

A similar result can be drawn in the case $p_0 > m$ (details are left to the reader). In particular,*each solution converges at equilibrium m when t* $\longrightarrow +\infty$. □

Example 6.1.2: Catenary Problem

A chain is suspended by two fixed points: What is the curve the hanging chain assumes under its own weight when supported only at its ends? In his Two New Sciences (1638), Galileo says that a hanging cord is an approximate parabola. The problem is as follows. What is this curve exactly?

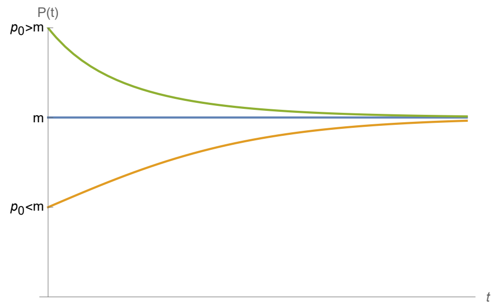


FIGURE 2. Solutions of the logistic equation.

SOL. — Let xy be the plane containing the curve; we use x as a parameter so that the curve is described as the graph of a function $\alpha = \alpha(x)$. Our goal is to determine this function. Let us see how the mechanics of the problem are used to determine a Differential Equation for α.

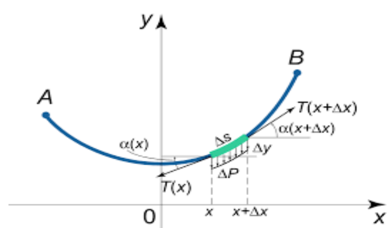


FIGURE 3. Catenary

Consider a small portion of chain delimited by points $(x, \alpha(x))$ and $(x + \delta x, \alpha(x + \delta x))$ ($\delta x > 0$ is "small"). In this part of the chain, the following forces act: *tension* exerted at two extremities by the remaining parts of the chain and *gravitation*. The last one is easy because it slopes downward as $m\vec{g}$. Here $\vec{g} = (0, -g)$ ($g = 9.8m/s^2$) while m is the mass of the small part of the chain. Let us say that $\varrho = \varrho(x)$ is the *linear mass density*, $m = \varrho \cdot ds$ where ds =length of the portion. By the Pythagorean Theorem

$$ds \approx \sqrt{(\delta x)^2 + (\alpha(x + \delta x) - \alpha(x))^2},$$

the approximation becoming more and more precise when $\delta x \approx 0$. In this case, $\alpha(x+\delta x)-\alpha(x) = \alpha'(x)\delta x+o(\delta x) \approx \alpha'(x)\delta x$, whence

$$m = \varrho(x)\sqrt{1 + \alpha'(x)^2}\delta x.$$

In conclusion

$$m\vec{g} = \left(0, -\varrho(x)g\sqrt{1 + \alpha'(x)^2}\delta x\right).$$

About the tension, let's denote by $\vec{T}(x)$ the force exercised by the part of the chain included between $(x, \alpha(x))$ and B (final point). Therefore, the force exercised by the part of the chain included between A and $(x, \alpha(x)))$ must be $-\vec{T}(x)$. Therefore, in $(x, \alpha(x))$ is acting $-\vec{T}(x)$, in $(x + \delta x, \alpha(x + \delta x))$ is acting $\vec{T}(x + \delta x)$ and these two are in equilibrium with $m\vec{g}$. This leads to the equation

$$-\vec{T}(x) + \vec{T}(x + \delta x) + m\vec{g} = \vec{0},$$

or, in components $\vec{T} = (\tau, \sigma)$,

$$\begin{cases} \tau(x+\delta x) - \tau(x) = 0, \\ \\ \sigma(x+\delta x) - \sigma(x) - \varrho(x) g \sqrt{1+\alpha'(x)^2}\delta x = 0. \end{cases}$$

The first equation says that $\tau(x) \equiv \tau_0$ is constant. In the second equation, dividing by δx and letting this to 0, we deduce

$$\sigma'(x) = \varrho(x) g \sqrt{1+\alpha'(x)^2}.$$

There is still one more information we need to use: $\vec{T}$ is tangent to the chain point by point. In particular, the angular coefficient of T must coincide with the angular coefficient of the tangent to y, namely, $\alpha'(x)$. Since $\vec{T} = (\tau_0, \sigma(x))$, we conclude

$$\frac{\sigma(x)}{\tau_0} = \alpha'(x), \quad \longrightarrow \quad \sigma(x) = \tau_0 \alpha'(x).$$

By this we obtain finally the following differential equation:

$$\alpha''(x) = \frac{g}{\tau_0} \varrho(x) \sqrt{1+\alpha'(x)^2}. \tag{6.1.3}$$

Since this equation involves the second derivatives of α, it is a second-order equation. However, it can be easily reduced to a first-order equation: setting $y(x) := \alpha'(x)$ we get

$$y'(x) = \frac{g}{\tau_0} \varrho(x) \sqrt{1+y(x)^2}. \tag{6.1.4}$$

This equation can be solved by a method similar to the one seems for the logistic equation. In this chapter, we will develop this method in general. Once y has been determined, one can calculate α knowing that $\alpha' = y$, that is, α is one of the primitives of y. □

Example 6.1.3: Newton's Equations

The most classical example of an ODE is the *Newton equation*, direct consequence of **Newton's second law**. A particle of mass m in movement under the effect of some force $\vec{F}$ satisfies

$$m\vec{a} = \vec{F},$$

where $\vec{a}$ is the acceleration of the particle. For simplicity, we assume that the mass is moving on a straight rail and can characterize its position in terms of a function $x = x(t)$, t representing time. Then $x'(t)$ represents the velocity, while $x''(t)$ is the acceleration. Furthermore, the force $\vec{F}$ can be identified with a scalar F. In general, physical forces depend on *position* $x(t)$ (as, for instance, in the case of *gravitational force* or *elastic force*), *velocity* $x'(t)$ (as in the case of *friction*) or directly by time t (if the intensity of the applied force changes in time). Therefore, Newton's second law assumes the form

$$mx''(t) = F(t, x(t), x'(t)).$$

A classical example is a mass m moving under the action of an elastic force and friction. If $\kappa \geqslant 0$ represents the elastic constant and assuming the origin as the rest position, elastic force is given by

$$-\kappa x(t).$$

The minus means that the force tends to move the particle back to the origin. Second component of applied force is friction, which depends on velocity. For simplicity, we will assume the rail be homogeneous and friction be proportional to velocity in a way to decelerate the mass. This means the force is

$$-\nu x'(t),$$

($\nu \geqslant 0$ is called *viscosity*). If an external force $f(t)$ (that is, independent of the mass) acts on the mass, the equation may be modified as

$$mx''(t) = -\kappa x(t) - \nu x'(t) + f(t).$$

With this simple equation we describe lots of phenomena like *forced oscillations*. An interesting and surprising example is the case of *resonance*. Imagine a periodic external force is applied to an harmonic oscillator,

$$mx''(t) = -\kappa x(t) + F_0 \sin(\omega t).$$

It turns out that if $\omega = \sqrt{\kappa}$ external force enters in resonance with elastic force leading to an x with oscillation amplitude increasing in time. A model like this was used to provide a simple explanation of the famous Takoma bridge collapse. □

6.2. First order linear equations

The first type of ODE we consider is the following

$$y'(t) = a(t)y(t) + b(t), \quad t \in I. \tag{6.2.1}$$

Malthus' equation (6.1.1) is an example of this type of equation ($a(t) \equiv r_0$, $b(t) \equiv 0$). Here, $a, b : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are known functions (called *coefficients*). If $b \equiv 0$ we say that the equation is *homogeneous*. In this case, the set of solutions has a *linear structure*. In fact, we can see that if φ and ψ are solutions, then any linear combination $\alpha\varphi + \beta\psi$ is also a solution (here $\alpha, \beta \in \mathbb{R}$). In fact

$$(\alpha\varphi + \beta\psi)'(t) = \alpha\varphi'(t) + \beta\psi'(t) = \alpha a(t)\varphi(t) + \beta a(t)\psi(t) = a(t)(\alpha\varphi + \beta\psi)(t), \quad t \in I.$$

Homogeneous equations are simpler to solve. We may adapt the argument user for Malthus' equation.

Proposition 6.2.1

Let $a \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ be an interval. Then, all the solutions of the homogeneous equation

$$y'(t) = a(t)y(t), \; t \in I,$$

are

$$y(t) = ce^{\int a(t)\,dt}, \; c \in \mathbb{R}. \tag{6.2.2}$$

Proof. Notice that $y \equiv 0$ is trivially a solution. If y is a solution in some interval I and $y(t_0) \neq 0$, at least in a suitable neighborhood of t_0 we have $y \neq 0$. Then

$$y'(t) = a(t)y(t) \iff \frac{y'(t)}{y(t)} = a(t), \iff (\log|y(t)|)' = a(t),$$

thus,

$$\log|y(t)| = \int a(t)\,dt + k,$$

where k is a constant. Therefore

$$|y(t)| = e^k e^{\int a(t)\,dt}, \iff y(t) = \pm e^k e^{\int a(t)\,dt} \equiv ce^{\int a(t)\,dt}, \; c \in \mathbb{R}\backslash\{0\}.$$

Notice that this y is never $= 0$, thus if $y(t_0) \neq 0$ at some t_0, then $y \neq 0$ always and y is provided by the previous formula. Since for $c = 0$ we obtain the null solution, the conclusion follows. □

Let's move to the general case of a non homogeneous equation,

$$y'(t) = a(t)y(t) + b(t).$$

We prove now that the general solution is obtained by summing to (6.2.2) a *particular solution* of the non homogeneous equation.

Proposition 6.2.2

Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. If $u = u(t)$ is a particular solution of the non homogeneous equation, then the general solution of

$$y'(t) = a(t)y(t) + b(t),\ t \in I,$$

is given by the formula

$$y(t) = ce^{\int a(t)\, dt} + u(t),\ t \in I. \tag{6.2.3}$$

Proof. Just note that y is a solution of the nonhomogeneous equation if and only if

$$(y-u)' = y' - u' = (ay+b) - (au+b) = a(y-u), \quad \overset{(6.2.2)}{\Longleftrightarrow} \quad y(t) - u(t) = ce^{\int a(t)\, dt}. \qquad \square$$

Thus, to determine the general solution for the non homogeneous equation it remains to determine a particular solution. This may be determined through the so called *method of variation of constants*:

Theorem 6.2.3

Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. Then, the general solution of

$$y'(t) = a(t)y(t) + b(t), \quad t \in I,$$

is

$$y(t) = e^{\int a(t)dt}\left[\int e^{-\int a(t)dt} b(t)\, dt + c\right], \quad t \in I, \tag{6.2.4}$$

where $c \in \mathbb{R}$ is a constant. Formula (6.2.4) is also called the **general integral** of the equation.

Proof. We start by looking for a particular solution $u = u(t)$. The idea is to look at the following.

$$u(t) = c(t)e^{\int a(t)\, dt}, \text{ where } c = c(t) \text{ is determined by imposition of } u' = au + b.$$

Now, being

$$u'(t) = \left(c(t)e^{\int a(t)\, dt}\right)' = c'(t)e^{\int a(t)\, dt} + c(t)e^{\int a(t)\, dt} a(t) = e^{\int a(t)\, dt}\left(c'(t) + a(t)c(t)\right),$$

we have

$$u' = au + b, \iff e^{\int a(t)\, dt}\left(c'(t) + a(t)c(t)\right) = a(t)c(t)e^{\int a(t)\, dt} + b(t),$$

that is

$$c'(t)e^{\int a(t)\, dt} = b(t), \iff c'(t) = e^{-\int a(t)\, dt} b(t), \iff c(t) = \int e^{-\int a(t)\, dt} b(t)\, dt + \widetilde{c},\ \widetilde{c} \in \mathbb{R}.$$

Thus

$$u(t) = \left(\int e^{-\int a(t)\, dt} b(t)\, dt + \widetilde{c}\right) e^{\int a(t)\, dt}$$

Plugging this formula into formula (6.2.3), we finally obtain

$$y(t) = ce^{\int a(t)\,dt} + \left(\int e^{-\int a(t)\,dt} b(t)\,dt + \widetilde{c}\right) e^{\int a(t)\,dt} = e^{\int a(t)\,dt}\left(\int e^{-\int a(t)\,dt} b(t)\,dt + c + \widetilde{c}\right),$$

and, absorbing the two constants $c, \widetilde{c}$ into a unique constant, we obtain formula (6.2.4) . □

EXAMPLE 6.2.4. *Find the general integral for the equation*

$$y'(t) - \frac{2}{t}y(t) = 1, \quad t \in]0, +\infty[.$$

SOL. — We have

$$y'(t) = \frac{2}{t}y(t) + 1 = a(t)y(t) + b(t), \quad \text{where } a(t) = \frac{2}{t},\ b(t) = 1.$$

Therefore

$$y(t) = e^{\int \frac{2}{t}dt}\left(\int e^{-\int \frac{2}{t}dt}dt + c\right) = e^{2\log t}\left(\int e^{-2\log t}dt + c\right) = t^2\left(\int \frac{1}{t^2}dt + c\right) = t^2\left(-\frac{1}{t} + c\right) = -t + ct^2. \quad \square$$

One should not be surprised because uniqueness does not hold for ODEs. Just the simplest among the differential equations, namely,

$$y' = 0,$$

has infinitely many solutions (all the constants). However, further conditions may lead to a unique solution. A very important case is the so called *Cauchy Problem* or *passage problem* or, again, *initial value* problem. This problem consists in finding a solution of an ODE fulfilling a passage/initial value condition. Formally, this problem may be stated in the following form:

$$(t_0, y_0) \begin{cases} y'(t) = a(t)y(t) + b(t), & t \in I, \\ y(t_0) = y_0. \end{cases} \tag{6.2.5}$$

Here, of course, $t_0 \in I$. It is easy to check that this problem has a unique solution:

Corollary 6.2.5

Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ an interval. Then, for every $t_0 \in I$, the Cauchy problem (t_0, y_0) admits a unique solution.

PROOF. Since the general solution is

$$y(t) = ce^{\int a(t)\,dt} + u(t) \equiv ce^{A(t)} + u(t), \text{ where } A(t) = \int a(t)\,dt,$$

we have that y solves (t_0, y_0) if and only if

$$y_0 = ce^{A(t_0)} + u(t_0), \iff c = \frac{y_0 - u(t_0)}{e^{-A(t_0)}}.$$

This c clearly exists ($e^{-A(t_0)} \neq 0$) and it is unique, thus we have existence and uniqueness for (t_0, y_0). □

EXAMPLE 6.2.6. *Solve the Cauchy Problem*

$$\begin{cases} y'(t) - \dfrac{2y(t)}{1-t^2} = t, \quad t > 1 \\ \\ y(2) = 0. \end{cases}$$

SOL. — Rewriting the equation in the canonical form

$$y'(t) = \frac{2}{1-t^2}y(t) + t, \implies y(t) = e^{\int \frac{2}{1-t^2}\,dt}\left(\int e^{-\int \frac{2}{1-t^2}\,dt} t\,dt + C\right).$$

Now

$$\int \frac{2}{1-t^2}dt = \int \frac{2}{(1-t)(1+t)}dt = \int \frac{1}{1-t} + \frac{1}{1+t}dt = -\int \frac{1}{t-1}dt + \log|1+t| = \log\left|\frac{t+1}{t-1}\right|.$$

Because $t \in]1, +\infty[$, $\frac{t+1}{t-1} > 0$, therefore

$$y(t) = e^{\log \frac{t+1}{t-1}}\left(\int e^{-\log \frac{t+1}{t-1}} t\,dt + C\right) = \frac{t+1}{t-1}\left(\int \frac{t-1}{t+1}t\,dt + C\right).$$

Now

$$\int t\frac{t-1}{t+1}\,dt = \int t\frac{t+1-2}{t+1}\,dt = \int t\,dt - 2\int \frac{t}{t+1}\,dt = \frac{t^2}{2} - 2\int\,dt + 2\int \frac{1}{t+1}\,dt = \frac{t^2}{2} - 2t + 2\log|t+1|,$$

and finally

$$\varphi(t) = \frac{t+1}{t-1}\left(\frac{t^2}{2} - 2t + 2\log(1+t) + C\right), \quad t \in]0, +\infty[.$$

Imposing $\varphi(2) = 0$ we have

$$2(2 - 4 + 2\log 3 + C) = 0, \iff C = 2(1 - \log 3). \quad \square$$

6.3. First order separable variables equations

Linear equations are easy: there is a general formula to represent all the possible solutions and that's all, it is just a matter of application of this formula. Adjective "linear" stands for the dependence of the r.h.s. of the equation on unknown y. For a linear equation this is a first degree polynomial in y.

Many important phenomena are described through *non-linear equations*. For example, this is the case of the *logistic equation* (6.1.2) or the *catenary equation* (6.1.4). Both belong to a general type of equations of the form

$$y'(t) = a(t)f(y(t))$$

called *separable variables equations*. For ease on notations, we write these equations also in the form

$$y' = a(t)f(y).$$

First order linear homogeneous equations are examples of separable variables equations:

$$y'(t) = a(t)y(t), \text{ when } f(y) := y.$$

Imitating the way we solved homogeneous linear equations, we may say that

$$y'(t) = a(t)f(y(t)), \iff \frac{y'(t)}{f(y(t))} = a(t).$$

This passage is called *separation of variables*, because we bring all terms containing the unknown y on one side, and all terms depending on t but not on y on the other side. Previous relation is a true equivalence provided

$f(y(t)) \neq 0$. Let us assume this verified. The key trick is now to look at the l.h.s. $\frac{y'(t)}{f(y(t))}$ as a derivative respect to t of some function. Let us see this with an example.

Example 6.3.1. *Solve the equation,*

$$y'(t) = 1 + y(t)^2.$$

Sol. — We have a separable variables equation $y'(t) = a(t)f(y(t))$ with $a(t) \equiv 1$ and $f(y) = 1 + y^2$. In this case, whatever is $y(t)$ we have $f(y(t)) = 1 + y(t)^2 \neq 0$, thus

$$y'(t) = 1 + y(t)^2, \iff \frac{y'(t)}{1 + y(t)^2} = 1,$$

is a true equivalence. Now, we may notice that

$$\frac{y'(t)}{1 + y(t)^2} = (\arctan y(t))',$$

thus the equation takes the equivalent form

$$(\arctan y(t))' = 1, \iff \arctan y(t) = \int 1 \, dt + c = t + c, \; c \in \mathbb{R}.$$

This equation contains still the unknown $y(t)$ in implicit form. Easily, the solution can be extracted:

$$\arctan y(t) = t + c, \iff y(t) = \tan(t + c), \; c \in \mathbb{R}.$$

Notice that these are all the possible solutions, namely what we usually call *general integral*. □

Example 6.3.2. *Solve the equation*

$$y'(t) = te^{y(t)}.$$

Sol. — Also in this case, we have a separable variables equation, $y' = a(t)f(y)$ with $f(y) = e^y$ and $a(t) = t$. As in the previous example, $f(y) \neq 0$, thus the equation is equivalent to

$$\frac{y'(t)}{e^{y(t)}} = t, \iff e^{-y(t)}y'(t) = t.$$

Now, since

$$e^{-y(t)}y'(t) = \left(-e^{-y(t)}\right)',$$

we conclude that

$$\left(-e^{-y(t)}\right)' = t, \iff -e^{-y(t)} = \frac{t^2}{2} + c, \iff y(t) = -\log\left(-\frac{t^2}{2} - c\right), \; c \in \mathbb{R}. \quad \square$$

The method of *separation of variables* can be discussed in general. Suppose we know $f(y(t)) \neq 0$. Until this is true,

$$y'(t) = a(t)f(y(t)), \iff \frac{y'(t)}{f(y(t))} = a(t).$$

Now, we look for a G such that

$$\frac{y'(t)}{f(y(t))} = (G(y(t)))'.$$

If this function G exists, then

$$\boxed{G(y(t)) = \int a(t) \, dt + c, \; c \in \mathbb{R}.} \tag{6.3.1}$$

This is the *implicit form* of the solution. Finally, if G can be inverted, we obtain the *explicit form* of the solution:

$$\boxed{y(t) = G^{-1}\left(\int a(t)\, dt + c\right),\ c \in \mathbb{R}.} \tag{6.3.2}$$

In conclusion, once G is known, the equation is solved. About G just notice that

$$G(y(t)) = \int \frac{y'(t)}{f(y(t))}\, dt \overset{v=y(t),\ dv=y'(t)\ dt}{=} \int \frac{1}{f(v)}\, dv.$$

Thus, $G(y(t))$ is the primitive of $\frac{1}{f(v)}$ with the replacement $v = y(t)$.

The previous argument fails if $f(y(t)) = 0$ at some t. Values y_0 where f vanishes, that is $f(y_0) = 0$, are particularly important for a separable variables equation:

Proposition 6.3.3

Let $a(t) \not\equiv 0$. Then, $y(t) \equiv y_0$ (constant/stationary solution) if and only if $f(y_0) = 0$.

Proof. We have, $y(t) \equiv y_0$ is a solution iff $0 = a(t)f(y_0)$, iff $f(y_0) = 0$. □

Thus: if $f(y(t)) \neq 0$ (always), the solution can be determined through separation of variables whereas, if $f(y(t)) \equiv 0$, solution is constant. These are the unique possibilities provided f is sufficiently regular:

Theorem 6.3.4: General integral

Assume $a \in \mathscr{C}$ and $f \in \mathscr{C}^1$ (that is, $f, f' \in \mathscr{C}$). Then, if $y = y(t)$ is a solution of the equation

$$y'(t) = a(t)f(y(t)),$$

we have

- either $f(y(t)) \neq 0$ always and the solution if given by formula (6.3.1) or (6.3.2);
- or $f(y(t)) \equiv 0$, and $y(t) \equiv y_0$, where $f(y_0) = 0$.

Proof. Omitted. □

Is this dichotomy always true? Answer is no, as the following example shows.

Example 6.3.5. *Solve the equation*

$$y' = y^{1/3}.$$

Sol. — We have a separable variables equation $y' = a(t)f(y)$, where $a(t) \equiv 1$, $f(y) = y^{1/3}$. Notice that while $f \in \mathscr{C}(\mathbb{R})$, $f'(y) = \frac{1}{3}y^{-2/3}$ is not even defined at $y = 0$. In particular, $f' \notin \mathscr{C}(\mathbb{R})$.

Notice that constant solutions $y(t) \equiv y_0$ are determined by $f(y_0) = 0$, in our case $y_0^{1/3} = 0$, that is $y_0 = 0$. The conclusion is: *there is a unique constant solution,* $y(t) \equiv 0$.

Let us consider now a non constant solution. Assuming $y^{1/3} \neq 0$, that is $y \neq 0$, we have

$$y' = y^{1/3}, \iff \frac{y'}{y^{1/3}} = 1, \iff y^{-1/3}y' = 1.$$

Now,

$$y^{-1/3}y' = \left(\frac{3}{2}y^{2/3}\right)',$$

thus
$$\frac{3}{2}y^{2/3} = \int 1\ dt + c = t + c,\ c \in \mathbb{R}.$$
This is the implicit form. Extracting y
$$y = \left(\frac{2}{3}(t+c)\right)^{3/2}.$$
For example, take $c = 0$, $y(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2}$. Notice that $y(0) = 0$, but y is non constant. □

Example 6.3.6 (Catenary). *Assuming constant mass density, solve the catenary equation (6.1.3)*

Sol. — Setting $v(x) =: y'(x)$ we have the first order equation
$$v' = \frac{\varrho g}{\tau_0}\sqrt{1+v^2},$$
which is a particular case of *separable variables equation*. Here $a(x) \equiv \frac{\varrho g}{\tau_0}$, $f(v) = \sqrt{1+v^2}$. Clearly $a \in \mathscr{C}(\mathbb{R})$ and $f \in \mathscr{C}^1(\mathbb{R})$. Notice also that $f \neq 0$ always. Thus, to determine solutions we can separate variables,
$$\frac{v'}{\sqrt{1+v^2}} = \frac{\varrho g}{\tau_0},\ \iff\ \int \frac{v'}{\sqrt{1+v^2}}\ dx = \frac{\varrho g}{\tau_0}x + \gamma.$$
Now,
$$\int \frac{v'}{\sqrt{1+v^2}}\ dt \stackrel{u=v(x)}{=} \int \frac{1}{\sqrt{1+u^2}}\ du = \sinh^{-1} u = \sinh^{-1} v(x).$$
In conclusion
$$\sinh^{-1} v(x) = \frac{\varrho g}{\tau_0}x + \gamma,\ \iff\ v(x) = \sinh\left(\frac{\varrho g}{\tau_0}x + \gamma\right).$$
Finally, because $v = y'$,
$$y(x) = \int \sinh\left(\frac{\varrho g}{\tau_0}x + \gamma\right)\ dx + \widetilde{\gamma} = \frac{\tau_0}{\varrho g}\cosh\left(\frac{\varrho g}{\tau_0}x + \gamma\right) + \widetilde{\gamma}.$$
For instance assume that $A = (-\ell/2, h)$, $B = (\ell/2, h)$. Parameters $\gamma, \widetilde{\gamma}$ are determined by solving
$$\begin{cases} \frac{\tau_0}{\varrho g}\cosh\left(-\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right) + \widetilde{\gamma} = h, \\ \\ \frac{\tau_0}{\varrho g}\cosh\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right) + \widetilde{\gamma} = h. \end{cases}$$
Taking the difference, $\cosh\left(-\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right) = \cosh\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right)$, and because cosh is even, the unique possibility if that $-\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma = -\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right)$ that is, $\gamma = 0$. Therefore, $\widetilde{\gamma} = h - \frac{\tau_0}{\varrho g}\cosh\frac{\ell}{2}\frac{\varrho g}{\tau_0}$, whence
$$y(x) = \frac{\tau_0}{\varrho g}\cosh\frac{\varrho g}{\tau_0}x + h - \frac{\tau_0}{\varrho g}\cosh\frac{\ell}{2}\frac{\varrho g}{\tau_0}. \quad \square$$
Let now discuss the Cauchy problem
$$CP(t_0, y_0)\ \begin{cases} y'(t) = a(t)f(y(t)), \\ \\ y(t_0) = y_0. \end{cases}$$
Here we need to express more precisely some technical details. We will assume that
$$a : I \subset \mathbb{R} \longrightarrow \mathbb{R},\ \ f : J \subset \mathbb{R} \longrightarrow \mathbb{R},$$

with I, J both intervals. Since the solution of the Cauchy problem must be defined at $t = t_0$, by the equation we need $y'(t_0) = a(t_0)f(y(t_0)) = a(t_0)f(y_0)$, thus $t_0 \in I$ and $y_0 \in J$ or, with another notation, $(t_0, y_0) \in I \times J$. Under good assumptions on f, the Cauchy problem has always a unique solution:

Theorem 6.3.7: Existence and uniqueness

Let $a \in \mathscr{C}(I)$, $f \in \mathscr{C}^1(J)$, $I, J \subset \mathbb{R}$ intervals. Then, for every passage condition (t_0, y_0), the $CP(t_0, y_0)$ has a unique solution.

PROOF. Omitted. □

If the regularity on f is lacking, we may well have non uniqueness, as the following example shows:

EXAMPLE 6.3.8. *The Cauchy problem*

$$\begin{cases} y' = y^{1/3}, \\ \\ y(0) = 0, \end{cases}$$

has (at least) two solutions.

SOL. — We analysed the differential equation $y' = y^{1/3}$ in Example 6.3.5. Clearly, $y \equiv 0$ is a solution of the Cauchy problem. Also $y(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2}$ solves the equation and since $y(0) = 0$, it fulfills the passage condition, thus y is a non constant solution of the same Cauchy problem. □

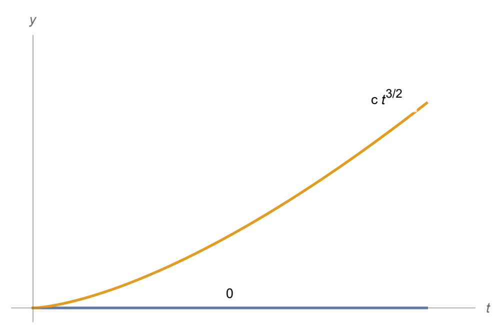


FIGURE 4. Non uniqueness: two different solutions of the same Cauchy problem.

Why uniqueness is so important? Imagine you have to do a prediction on the behaviour of a system whose evolution is described through a differential equation. You know the today state and you want to forecast the tomorrow state. If you have uniqueness, there is a unique possible future, thus the prediction works. But if uniqueness is lacking, then there is no hope to use solutions to do a prediction.

6.4. Second Order Linear Equations

We now consider equations of type

$$y''(t) = a(t)y'(t) + b(t)y(t) + f(t). \tag{6.4.1}$$

If $f \equiv 0$, the equation is called **homogenous** and if $a(t) \equiv a$, $b(t) \equiv b$ the equation is said to have *constant coefficients*. For simplicity, we will limit ourselves to this case, which we will rewrite as

$$y'' + ay' + by = f(t).$$

To begin with, we will consider the homogeneous case

$$y''(t) + ay'(t) + by(t) = 0.$$

To solve this equation in general, we perform the following trick. Denote by D the derivative, then the previous equation can be rewritten as

$$(D^2 + aD + b)y = 0.$$

The polynomial

$$\lambda^2 + a\lambda + b$$

is called **characteristic polynomial** and basically contains all the information to look for solutions.

Theorem 6.4.1

The general integral of $y'' + ay' + by = 0$ is

$$c_1 w_1(t) + c_2 w_2(t), \quad c_1, c_2 \in \mathbb{R},$$

where

- if $\Delta = a^2 - 4b > 0$, $(w_1, w_2) = (e^{\lambda_1 t}, e^{\lambda_2 t})$ with $\lambda_{1,2}$ are the roots of the char. pol.;
- if $\Delta = 0$, $(w_1, w_2) = (e^{\lambda_1 t}, te^{\lambda_1 t})$ with λ_1 is the unique root of the char. pol.;
- if $\Delta < 0$, $(w_1, w_2) = (e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t))$ with $\lambda_{1,2} = \alpha \pm i\beta$ are the complex roots of the char. pol.

The couple (w_1, w_2) is called **fundamental system of solutions**.

Proof. We consider three cases: $\Delta > 0$, $\Delta = 0$, $\Delta < 0$.

Case $\Delta > 0$: the characteristic polynomial can be factorized as

$$\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2),$$

hence

$$D^2 + aD + b = (D - \lambda_1)(D - \lambda_2).$$

Therefore,

$$(D^2 + aD + b)y = 0, \quad \Longleftrightarrow \quad (D - \lambda_1)(D - \lambda_2)y = 0.$$

Now call $\psi = (D - \lambda_2)y$. Then

$$(D - \lambda_1)\psi = 0, \quad \Longleftrightarrow \quad \psi' = \lambda_1 \psi, \quad \Longleftrightarrow \quad \psi = ce^{\lambda_1 t}.$$

But then

$$(D - \lambda_2)y = c_1 e^{\lambda_1 t}, \quad \Longleftrightarrow \quad y' = \lambda_2 y + c_1 e^{\lambda_1 t}.$$

This is a first order linear equation that may be easily solved by the general formula (6.2.4), obtaining

$$y(t) = e^{\lambda_2 t}\left(\int e^{-\lambda_2 t} c_1 e^{\lambda_1 t}\, dt + c_2\right) = e^{\lambda_2 t}\left(c_1 \int e^{(\lambda_1 - \lambda_2)t}\, dt + c_2\right) = \frac{c_1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

and being c_1, c_2 arbitrary, we get finally

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Case $\Delta = 0$: we can repeat the same computations as before, just to the point

$$y(t) = e^{\lambda_2 t}\left(c_1 \int e^{(\lambda_1-\lambda_2)t}\, dt + c_2\right),$$

but now $\lambda_1 = \lambda_2$, therefore

$$y(t) = e^{\lambda_1 t}\left(c_1 \int \, dt + c_2\right) = c_1 t e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Case $\Delta < 0$: Calculations of case $\Delta > 0$ can be repeated literally, leading to the same formula but with $\lambda_1, \lambda_2 \in \mathbb{C}$. Since coefficients $a, b, c \in \mathbb{R}$, $\lambda_{1,2} = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$, and $e^{\lambda_{1,2} t} = e^{\alpha t}(\cos(\beta t) \pm i \sin(\beta t))$. Since the equation is linear, also $\frac{1}{2}(e^{\lambda_1 t} + e^{\lambda_2 t}) = e^{\alpha t}\cos(\beta t)$ and $\frac{1}{2i}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) = e^{\alpha t}\sin(\beta t)$ are solutions, and from this the conclusion follows. □

As for linear first order equations, the general solution can be obtained by the general solution of the homogeneous equation by adding a particular solution:

Proposition 6.4.2

Let (w_1, w_2) be a fundamental system of solutions for the homogeneous equation

$$y'' + ay' + by = 0, \tag{6.4.2}$$

and u be a particular solution of the equation,

$$y'' + ay' + by = f(t). \tag{6.4.3}$$

Then, the general integral of (6.4.3) is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t), \quad c_1, c_2 \in \mathbb{R}.$$

Proof. Just note that if y solves (6.4.3), then $y - u$ solves (6.4.2). Indeed

$$(y-u)'' + a(y-u)' + b(y-u) = y'' + ay' + by - (u'' + au' + bu) = f - f = 0, \implies y - u = c_1 w_1 + c_2 w_2.$$ □

To complete the solution of the second-order nonhomogeneous equation, we need to determine a particular solution. As for the first-order case, this can be determined using the *method of variation of constants*. We look at u of type

$$u(t) = c_1(t) w_1(t) + c_2(t) w_2(t), t \in I.$$

searching for the coefficients c_1, c_2 in such a way that u be a solution of the equation.

Theorem 6.4.3: Lagrange

Let (w_1, w_2) a fundamental system of solutions of (6.4.2). Define

$$W(t) := \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix},$$

the **wronskian** of (w_1, w_2). Then, $W(t) \neq 0$ for all t and

$$u(t) = -\left(\int \frac{w_2(t)}{W(t)} f(t)\, dt \right) w_1(t) + \left(\int \frac{w_1(t)}{W(t)} f(t)\, dt \right) w_2(t), \quad t \in I, \tag{6.4.4}$$

is a particular solution of (6.4.3).

Proof. Let $u = c_1 w_1 + c_2 w_2$ with $c_j \equiv c_j(t)$ $j = 1, 2$. Then

$$u' = c_1' w_1 + c_1 w_1' + c_2' w_2 + c_2 w_2'.$$

To simplify the computations, we impose the condition

$$c_1' w_1 + c_2' w_2 = 0.$$

Then

$$u'' = c_1' w_1' + c_1 w_1'' + c_2' w_2' + c_2 w_2''.$$

Hence

$$u'' = au' + bu + f, \iff c_1' w_1' + c_2' w_2' = f.$$

We may conclude that u is a solution iff

$$\begin{cases} c_1' w_1 + c_2' w_2 = 0, \\ c_1' w_1' + c_2' w_2' = f. \end{cases} \tag{6.4.5}$$

This can be seen as a 2×2 linear system in the unknown (c_1', c_2') and the coefficients of the matrix

$$\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}.$$

Denoting with $W(t)$ the determinant of the previous matrix,

$$W(t) := w_1 w_2' - w_2 w_1', \quad (\textbf{wronskian of } (w_1, w_2))$$

it is easy to check that in all cases $W(t) \neq 0$ for any t:

$$\det \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix} = e^{(\lambda_1 + \lambda_2)t} (\lambda_2 - \lambda_1). \quad \det \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ \lambda e^{\lambda t} & (1 + \lambda t) e^{\lambda t} \end{bmatrix} = e^{2\lambda t},$$

and

$$\det \begin{bmatrix} e^{\alpha t} \cos(\beta t) & e^{\alpha t} \sin(\beta t) \\ e^{\alpha t}(\alpha \cos(\beta t) - \beta \sin(\beta t)) & e^{\alpha t}(\alpha \sin(\beta t) + \beta \cos(\beta t)) \end{bmatrix} = \beta e^{2\alpha t}.$$

Therefore,

$$c_1'(t) = \frac{-w_2(t) f(t)}{W(t)}, \quad c_2'(t) = \frac{w_1(t) f(t)}{W(t)}.$$

that is

$$c_1(t) = -\int \frac{w_2(t)}{W(t)} f(t)\, dt, \quad c_2(t) = \int \frac{w_1(t)}{W(t)} f(t)\, dt. \tag{6.4.6}$$

So, we get the formula (6.4.4). □

EXAMPLE 6.4.4. *Find the general integral of the equation*

$$y''(t) + y'(t) - 6y(t) = 2e^{-t}, \ t \in \mathbb{R}.$$

SOL. — We start by computing the fundamental system of solutions of the homogeneous equation. The characteristic polynomial is

$$\lambda^2 + \lambda - 6 = 0, \quad \Delta = 1 + 24 = 25 > 0, \quad \lambda_\pm = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Therefore the fundamental solutions are $w_1(t) = e^{2t}$, $w_2(t) = e^{-3t}$ with wronskian

$$W(t) = (-3-2)e^{-t} = -5e^{-t}.$$

By Lagrange formula (6.4.4), we have

$$\begin{aligned} u(t) \quad &= -\int \frac{e^{-3t}}{-5e^{-t}} 2e^{-t}\, dt\, e^{2t} + \int \frac{e^{2t}}{-5e^{-t}} 2e^{-t}\, dt\, e^{-3t} = \frac{2}{5}\int e^{-3t}\, dt\, e^{2t} - \frac{2}{5}\int e^{2t}\, dt\, e^{-3t} \\ &= -\frac{2}{15}e^{-t} - \frac{2}{10}e^{-t} = -\frac{1}{3}e^{-t}. \end{aligned}$$

Therefore, the general integral is

$$y(t) = c_1 e^{2t} + c_2 e^{-3t} - \frac{1}{3}e^{-t}, \quad c_1, c_2 \in \mathbb{R}. \quad \square$$

In the case of second-order equations, we see an interesting phenomenon: the general integral depends on two free constants c_1, c_2. Therefore, it is clear that a unique condition in y is not sufficient to determine a unique solution. That is why the Cauchy problem for a second-order ODE is different with respect to the first-order case. Intuitively, we need a second condition for the solution. There are two interesting cases:

- The **Cauchy problem**, which consists of finding a solution y that satisfies two initial conditions, as

$$CP(t_0, y_0, y_0') \begin{cases} y'' + ay' + by = f(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_0'. \end{cases}$$

- the **boundary value problem**, that consists in finding a solution y fulfilling two passage conditions as

$$\mathrm{BV}(t_0, y_0; t_1, y_1) \begin{cases} y'' + ay' + by = f(t), \\ y(t_0) = y_0, \\ y(t_1) = y_1. \end{cases}$$

These are two entirely different problems. Thinking to the differential equation as coming from the Newton law $ma = F$ and y representing the position in function of time, in the first case the solution is the trajectory of motion starting at time t_0 at point y_0 with velocity y_0'. In the second case, the solution is the trajectory of motion starting at time t_0 at point y_0 and reaching point y_1 at time t_1. The different nature of these problems is reflected by different results we may prove concerning existence and uniqueness. This holds true in general for the Cauchy Problem, but might be false for the Boundary Value Problem.

Theorem 6.4.5: existence and uniqueness

The Cauchy Problem $CP(t_0, y_0, y_0')$ has a unique solution for any $t_0 \in I$ and $y_0, y_0' \in \mathbb{R}$.

Proof. We have to prove that there exists a unique c_1, c_2 such that

$$y = c_1 w_1 + c_2 w_2 + u,$$

is a solution of $CP(t_0, y_0, y_0')$. Imposing the two initial conditions we have,

$$\begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) + u(t_0) = y_0, \\ \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) + u'(t_0) = y_0', \end{cases} \iff \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) = y_0 - u(t_0), \\ \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) = y_0' - u'(t_0). \end{cases}$$

This last as a 2×2 linear system whose coefficients matrix is the wronskian matrix. Since in our assumption the wronskian matrix is invertible, previous system has a unique solution c_1, c_2. $\square$

Example 6.4.6. *Solve the Cauchy Problem*

$$\begin{cases} y''(t) + y(t) = e^t, \; t \in \mathbb{R}, \\ y(0) = 0, \\ y'(0) = 1. \end{cases}$$

Sol. — The characteristic equation is $\lambda^2 + 1 = 0$, that is $\lambda = \pm i$. Therefore $w_1(t) = \cos t$, $w_2(t) = \sin t$ is a fundamental system of solutions for the homogenous equation. The wronskian is

$$W(t) = \det \begin{bmatrix} \cos t & \sin t \\ \\ -\sin t & \cos t \end{bmatrix} = (\cos t)^2 + (\sin t)^2 = 1.$$

Therefore a particular solution, by the Lagrange formula, is

$$\begin{aligned} u(t) &= -\left(\int \frac{\sin t}{1} e^t \, dt \right) \cos t + \left(\int \frac{\cos t}{1} e^t \, dt \right) \sin t = -\left(\int e^t \sin t \, dt \right) \cos t + \left(\int e^t \cos t \, dt \right) \sin t \\ &= -\frac{e^t}{2} (\sin t - \cos t) \cos t + \frac{e^t}{2} (\cos t + \sin t) \sin t = \frac{e^t}{2}. \end{aligned}$$

Hence the general integral is

$$\varphi(t) = c_1 \cos t + c_2 \sin t + \frac{e^t}{2}.$$

Now, imposing the initial conditions we get the system

$$\begin{cases} c_1 + \frac{1}{2} = 0, \\ \\ c_2 + \frac{1}{2} = 1, \end{cases} \iff c_1 = -\frac{1}{2}, \; c_2 = \frac{1}{2}, \implies \varphi(t) = \frac{1}{2} \left(\sin t - \cos t + e^t \right). \quad \square$$

6.4.1. Applications.

Example 6.4.7 (Damped Oscillations). *Describe the motion of a point mass moving on a straight line under the action of elastic force and friction.*

Sol. — Let m be the mass, $\kappa > 0$ the elastic constant and $\nu > 0$ the viscosity ν. If $y = y(t)$ is the position at time t

$$my''(t) = -\kappa y(t) - \nu y'(t).$$

The equation is a second order linear equation with constant coefficients. Its characteristic equation is

$$m\lambda^2 + \nu\lambda + \kappa = 0.$$

Because $\Delta = \nu^2 - 4m\kappa$ we have that if $\Delta \geqslant 0$, that is if $\nu^2 \geqslant 4m\kappa$, $\nu \geqslant \sqrt{4m\kappa}$, the general solution of the equation is,

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{where } \lambda_{1,2} = \frac{-\nu \pm \sqrt{\Delta}}{2m}.$$

Since $\Delta > \nu^2$, $-\nu \pm \sqrt{\Delta} < -\nu + \sqrt{\nu^2} = 0$, that is $\lambda_{1,2} < 0$. The mass is exponentially "attracted" from the origin. The case $\Delta = 0$ is analogous.

When $\Delta < 0$, we have

$$\lambda_{1,2} = -\frac{\nu}{2m} \pm i\frac{\sqrt{-\Delta}}{2m},$$

and the general solutions of the equation is

$$y(t) = c_1 e^{-\frac{\nu}{2m}t} \cos\left(\frac{\sqrt{-\Delta}}{2m}t\right) + c_2 e^{-\frac{\nu}{2m}t} \sin\left(\frac{\sqrt{-\Delta}}{2m}t\right).$$

The motion is oscillatory (because of sin and cos), the oscillations being damped by the exponential $e^{-\frac{\nu}{2m}t}$. □

Example 6.4.8 (Resonance). *A point mass is subject to the action of an elastic force (elastic constant k^2) and to and external time periodic force proportional to* $\sin(kt)$. *Discuss the behaviour of the system in the case*

Sol. — Assuming unitary mass and force, we can describe the system through the equation

$$y''(t) = -k^2 y(t) + \sin(kt). \tag{6.4.7}$$

The characteristic equation is $\lambda^2 = -k^2$, that is, $\lambda = \pm ik$, therefore, the fundamental system of solutions for homogeneous equations is $w_1(t) = \cos(kt)$, $w_2(t) = \sin(kt)$. The wronskian is $W(t) \equiv k$ and a particular solution is

$$u(t) = -\left(\int \frac{\sin(kt)}{k}\sin(kt)\,dt\right)\cos(kt) + \left(\int \frac{\cos(kt)}{k}\sin(kt)\,dt\right)\sin(kt).$$

Now

$$\begin{aligned}\int \sin(kt)^2\,dt &= \int \sin(kt)\sin(kt)\,dt = -\frac{1}{k}\int \sin(kt)\,(\cos(kt))' = -\frac{1}{k}\left[\sin(kt)\cos(kt) - k\int \cos(kt)^2\,dt\right]\\ &= -\frac{1}{2k}\sin(2kt) + \int (1 - \sin(kt)^2)\,dt = -\frac{1}{2k}\sin(2kt) + t - \int \sin(kt)^2\,dt\end{aligned}$$

and by this we have

$$\int \sin(kt)^2\,dt = \frac{t}{2} - \frac{\sin(2kt)}{4k}.$$

Moreover

$$\int \frac{\cos(kt)}{k}\sin(kt)\,dt = \frac{1}{2k}\int \sin(2kt)\,dt = -\frac{\cos(2kt)}{4k^2}.$$

In conclusion

$$u(t) = \left(\frac{\sin(2kt)}{4k^2} - \frac{t}{2k}\right)\cos(kt) - \frac{\cos(2kt)}{4k^2}\sin(kt).$$

Here we may notice that u is unbounded. This fact has been used to explain the impressive collapse of the Tacoma bridge. □

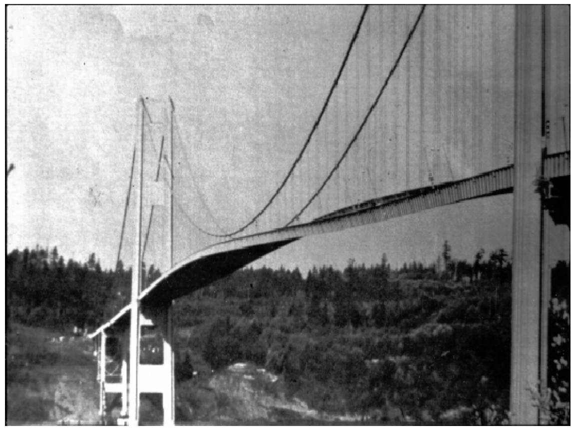


FIGURE 5. Tacoma bridge collapse

6.5. Exercises

EXERCISE 6.5.1. Find the general integral of the following equations:

1. $y' + (\cos t)y = \frac{1}{2}\sin(2t),\ t \in \mathbb{R}$.
2. $y' - \frac{t}{1-t^2}y = t,\ t \in]-1,1[$.
3. $y' + 2ty = 2t^3,\ t \in \mathbb{R}$.
4. $y' - \frac{1}{t}y + \frac{\log t}{t} = 0,\ t \in]0,+\infty[$.
5. $y' - (\tan t)y = t^3,\ t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
6. $y' + 2ty = te^{-t^2},\ t \in \mathbb{R}$.
7. $y' + y = \sin t,\ t \in \mathbb{R}$.
8. $y' + (\cos t)y = (\cos t)^2,\ t \in \mathbb{R}$.
9. $y' = \frac{2t}{t^2+1}y + 2t(t^2+1),\ t \in \mathbb{R}$.

EXERCISE 6.5.2. Solve the Cauchy Problem

$$\begin{cases} y'(t) + \frac{3t^2}{t^3+5}y(t) = \sqrt[3]{t}, \\ y(0) = 1. \end{cases}$$

EXERCISE 6.5.3. Consider the equation

$$y' - (\tan t)y = \frac{1}{\sin t}, \quad t \in \left]0, \frac{\pi}{2}\right[.$$

i) Find its general integral. ii) Is it true that for every solution it holds $\lim_{t\to 0+} y(t) = -\infty$? iii) Are there solutions such that $\exists \lim_{t\to\frac{\pi}{2}-} y(t) \in \mathbb{R}$. In the affirmative case, what is the value of the limit?

EXERCISE 6.5.4. Consider the equation

$$y' + (\sin t)y = \sin t, \quad t \in \mathbb{R}.$$

i) Find its general integral. ii) Are there solutions y such that $\exists \lim_{t\to+\infty} y(t) \in \mathbb{R}$. iii) Find the solution of the Cauchy Problem $y\left(\frac{\pi}{2}\right) = 1$.

Exercise 6.5.5. Consider the equation

$$y'(t) = -\frac{1}{t} y(t) + \arctan t.$$

Find its general integral on $]-\infty, 0[$ and on $]0, +\infty[$. Does it exists a $y : \mathbb{R} \longrightarrow \mathbb{R}$ solution on both $]-\infty, 0[$ and $]0, +\infty[$. In this case, what is $y(0)$?

Exercise 6.5.6. Solve the Cauchy problems

1. $\begin{cases} y' = \dfrac{y^2 - y - 2}{3} \arcsin t, \\ y(0) = 3. \end{cases}$ 2. $\begin{cases} y' = \dfrac{y(2y-1)}{\cosh t}. \\ y(0) = 1. \end{cases}$ 3. $\begin{cases} y' = \dfrac{\cos^2(2y)}{t(2 - \log^2 t)} \\ y(1) = \frac{\pi}{2}. \end{cases}$ 4. $\begin{cases} y' = \dfrac{(e^t + 1) y^2}{e^t + 2}, \\ y(0) = 1/2. \end{cases}$

Exercise 6.5.7. Solve, in function of the initial condition $y(0) = y_0$, the Cauchy problem

$$\begin{cases} y' = 4y(1 - y), \\ y(0) = y_0. \end{cases}$$

How many typical plots are there for the solutions?

Exercise 6.5.8. Consider the Cauchy problem

$$\begin{cases} y' = y(1 - y^2), \\ y(0) = 1/2. \end{cases}$$

Determine the implicit form for the solution and, if possible, the explicit form. Is it true that the solution is defined for all times $t \in \mathbb{R}$? Justify carefully.

Exercise 6.5.9. For each of the following equations determine a fundamental system of solutions and write the general integral.

1. $y'' - 3y' + 2y = 0$. 2. $y'' - 2y' + 2y = 0$. 3. $y'' - 4y + 3y = 0$. 4. $y'' + y' = 0$. 5. $y'' - y' + y = 0$.

Exercise 6.5.10. Determine the general integral of the following equations:

1. $y''(t) + y'(t) - 6y(t) = 2e^{-t}$. 2. $y'' - y' + y = e^t$. 3. $y'' + 4y' + 2y = t^2$. 4. $y'' + 2y' = e^t$.

5. $y'' - y = \cos t$. 6. $y'' + y = \frac{1}{\cos t}$. 7. $y'' + 2y' + 2y = 2t + 3 + e^{-t}$. 8. $y'' - 2y' + 2y = e^t \cos t$.

Exercise 6.5.11. For each of the following equations find the general integral and the solution of the Cauchy Problem with initial conditions $y(0) = y'(0) = 0$.

1. $y'' - y = t$. 2. $y'' + 4y = e^t$. 3. $y'' + y = t$.

4. $y'' + y' - 6y = -4e^t$. 5. $y'' - 8y' + 17y = 2t + 1$. 6. $y'' + y = \frac{1}{\cos t}$.

Exercise 6.5.12. Consider the differential equation

$$y''(t) - y'(t) = te^t, \quad t \in \mathbb{R}.$$

i) Compute its general integral. ii) Are there solutions such that $\lim_{t \to +\infty} y(t) \in \mathbb{R}$? iii) Determine the solution of the Cauchy Problem $y(0) = 1$, $y'(0) = 0$.

EXERCISE 6.5.13. Determine the general integral of the equation

$$y''(t) - 5y'(t) - 6y(t) = 16e^{-2t},\ t \in \mathbb{R}.$$

Hence, say if there exists a solution such that $y(0) = 0$ and $\lim_{t\to+\infty} y(t) = 0$.

EXERCISE 6.5.14. Consider the equation

$$y''(t) + y'(t) = t + \cos t, \quad t \in \mathbb{R}.$$

i) Determine its general integral. ii) Are there solutions y such that $\exists \lim_{t\to+\infty} y(t) \in \mathbb{R}$? iii) Are there solutions y such that $y(0) = 0$ and $\lim_{t\to-\infty} y(t) = +\infty$.

EXERCISE 6.5.15. Consider the equation

$$y''(t) + y(t) = \frac{1}{\cos t}, \quad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

i) Determine its general integral. ii) Is it true that every solution of the equation fulfills $\lim_{t\to\frac{\pi}{2}-} y(t) = +\infty$? iii) Are there solutions such that, for some $C \neq 0$, fulfils $y(t) \sim Ct^2$ when $t \to 0$.

EXERCISE 6.5.16. Find the general integral of the equation

$$y''(t) + 4y'(t) + 4 = \frac{e^{-2t}}{t^2},\ t \in]0, +\infty[.$$

Are there solutions of the equation such that $\exists \lim_{t\to 0+} y(t)$?

EXERCISE 6.5.17. A radioactive material decays of 20% in 10 days. Find his halving time.

EXERCISE 6.5.18. In an hospital, a radioactive substance is accumulated into a vessel at rate of 2m^3 each month. The radioactivity has a decay rate estimated to be proportional to the quantity present in the vessel according a constant of proportionality $k = -1$. Knowing that initially the vessel is empty find the total amount of radioactive substance contained when the vessel is full.

EXERCISE 6.5.19. The queue created by a car accident on an highway reduces at some rate inversely proportional to the square root of the length of the queue. Knowing that to halve a queue of 1km it takes 10min, how long it takes to halve a queue of 2km? How long it takes to eliminate the queue?

EXERCISE 6.5.20. In a fish breeding the population of fish is assumed to follows a logistic evolution

$$y'(t) = 0.1y(t) - by(t)^2,$$

where b is to be determined. You know that initially there is 500kg of fish. After one year, the fish has grown to 1.250kg. Determine the value of b.

EXERCISE 6.5.21 (★). A particle of mass m falls under action of gravity and air friction in such a way that the equation of motion is

$$ma(t) = -mg - mkv(t).$$

Express the velocity v as function of the quote x and determine $v = v(x)$ explicitly. What if friction is proportional to the square of v?

EXERCISE 6.5.22 (★★). A swimmer aims to cross a river of width ℓ. The starting point and the arriving point are aligned and orthogonal to the river. The water flows at constant speed v higher than the speed V of the swimmer. Assume the swimmer points, at every moment, its final destination. Is it possible to determine e path in such a way the swimmer reaches the other side of the river? (hint: use differential equations to describe the trajectory of the swimmer. . .).

Exercise 6.5.23 (★). A ship of mass m moves from rest under a constant propelling force mf and against a resistance mkv^2. Determine the speed $v = v(a)$ as function of the covered distance a. Suppose that, at certain a fixed, the engines are shut down. What is the distance needed to stop the ship?

Exercise 6.5.24. A mass m is vertically attached to two springs, with same elastic constant k. Initially the mass is at rest positions for the springs. Determine the motion of the mass, considering also the gravity.

CHAPTER 7

Advanced Differential Equations

In the previous Chapter we introduced to some basic types of differential equations. The focus has been on methods to explicitly solve equations. However, as for algebraic equations, there is a limited class of solvable equations. This motivates new method and tools for general equations. We need to answer to the following questions:

- How to ensure existence of solutions when they cannot be explicitly computed?
- What information can be drawn for solutions that cannot be explicitly computed? For instance, can we plot the diagram of the solution?

To answer these questions, two type of methods have been developed: *qualitative methods* and *quantitative/numerical methods*. In this Course we will develop tools for the former while numerical methods will be introduced in Numerical Calculus courses. Qualitative methods have the goal of providing some graphical representation of solutions, pretty much how we plot the graph of a function starting with its analytical expression.

Chapter requirements: Chapter on Basic Differential Equations, notably separable variables equations, a good comprehension of ordinary derivative and of its geometrical properties.

7.1. Scalar Equations

The *Cauchy problem for first order scalar equations* consists in finding a solution of a differential equation fulfilling a *passage* or *initial value* condition. Formally,

$$CP(t_0, y_0) \; : \; \begin{cases} y' = f(t, y), \quad t \in I, \\ \\ y(t_0) = y_0. \end{cases} \qquad \text{where } f = f(t, y) : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}.$$

A function $y = y(t) \in \mathscr{C}^1(I)$, $I \subset \mathbb{R}$ interval is a **solution** if it fulfils the equation and the passage/initial condition. Notice that, in particular:

$$(t, y(t)) \in D, \; \forall t \in I.$$

In other words, *the graph of the solution must be contained in the domain D of f.* In particular, because $(t_0, y(t_0)) = (t_0, y_0)$, the passage point $(t_0, y_0) \in D$.

The first problem is: *under which conditions on f* **there exists a unique solution** *to the Cauchy problem?* These question contains two sub-questions: when a solution **exists** and, provided it exists, when it is **unique**.

Existence holds under very general assumption on the structure of the equation, namely on the function $f = f(t, y)$:

Theorem 7.1.1: Peano

et $f = f(t, y) : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be **continuous** on D. Then, there exists a solution of $CP(t_0, y_0)$ for every passage condition $(t_0, y_0) \in D$.

Peano's theorem is a very weak result. Indeed, it does not ensures uniqueness which is a very important property when we use solutions of ODEs to do predictions on the behavior of a certain real world system. As Example 6.3.8 shows, in general uniqueness does not hold. In that Example,

$$y' = f(t, y), \quad \text{with } f(t, y) = y^{1/3} \in \mathscr{C}(\mathbb{R} \times \mathbb{R}).$$

Thus, *continuity of f is not sufficient to ensure uniqueness*. We need more on f to obtain also uniqueness.

There are two types of existence and uniqueness results: **global** and **local** theorems. With **global** we mean results that provides explicitly the information on the domain of definition of the solution. Vice versa, **local** results do not give any information on the consistence of the time interval of definition for the solution. We only know, a priori, that the solution is defined on some interval I containing initial time t_0 and nothing more. Global theorems are more restrictive but also easier to state and use. Conversely, local theorems are more general but a bit harder to use. Here we will limit to a statement per type, starting with the

Theorem 7.1.2: Global Cauchy–Lipschitz

Let $f : D = [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathscr{C}([a, b] \times \mathbb{R})$;

ii) $\partial_y f$ be bounded on $[a, b] \times \mathbb{R}$, that is

$$\exists L > 0 \;:\; |\partial_y f(t, y)| \leqslant L, \; \forall (t, y) \in [a, b] \times \mathbb{R}.$$

Then, for every $(t_0, y_0) \in [a, b] \times \mathbb{R}$ there exists a unique solution to $CP(t_0, y_0)$.

Example 7.1.3. *Check the global Cauchy–Lipschitz conditions for the equation*

$$y' = \frac{\sin(ty)}{1 + y^2}.$$

on any strip $[a, b] \times \mathbb{R}$.

Sol. — Here $f(t, y) = \frac{\sin(ty)}{1+y^2}$ is clearly defined on $(t, y) \in \mathbb{R} \times \mathbb{R}$ and $f \in \mathscr{C}$. Moreover

$$\partial_y f(t, y) = \frac{(1 + y^2)t \cos(ty) - 2y \sin(ty)}{(1 + y^2)^2} \in \mathscr{C}(\mathbb{R} \times \mathbb{R}).$$

Therefore

$$|\partial_y f(t, y)| = \left| \frac{(1 + y^2)t \cos(ty) - 2y \sin(ty)}{(1 + y^2)^2} \right| \leqslant \frac{(1 + y^2)|t| + 2|y|}{(1 + y^2)^2} = \frac{|t|}{1 + y^2} + \frac{2|y|}{(1 + y^2)^2}.$$

Clearly $\frac{1}{1+y^2} \leqslant 1$ and because $2ab \leqslant a^2 + b^2$, $\frac{2|y|}{(1+y^2)^2} \leqslant \frac{1+y^2}{(1+y^2)^2} = \frac{1}{1+y^2} \leqslant 1$, we have

$$|\partial_y f(t, y)| \leqslant |t| + 1 \leqslant \max\{|a|, |b|\} + 1 =: L, \; \forall (t, y) \in [a, b] \times \mathbb{R}. \qquad \square$$

The two main restriction of the Global Cauchy–Lipschitz Theorem are: first, *the domain that must be a strip* $[a, b] \times \mathbb{R}$; second, *the $\partial_y f$ that must be bounded.* If one or both these conditions are not fulfilled by f (we consider the continuity of f a minimal assumption), we cannot apply this result. The conclusions of the Global CL Theorem, can be false.

Example 7.1.4. *Solve the Cauchy problem*

$$\begin{cases} y' = y^2, \\ y(t_0) = y_0. \end{cases}$$

Sol. — Let $f(t,y) = y^2$. Clearly $f \in \mathscr{C}(\mathbb{R}\times\mathbb{R})$. However, $\partial_y f(t,y) = 2y$ is unbounded on every strip $[a,b]\times\mathbb{R}$, so the Global CL Thm doesn't apply. We can however solve the equation by separation of variables: noticed that $y \equiv 0$ is a solution, if $y \not\equiv 0$ we have

$$y'(t) = y(t)^2, \iff \frac{y'(t)}{y(t)^2} = 1, \iff \left(-\frac{1}{y(t)}\right)' = 1, \iff -\frac{1}{y(t)} = t + C, \iff y(t) = -\frac{1}{t+C}.$$

Notice that this solution is never $= 0$ therefore, the previous argument shows that if $y \neq 0$ somewhere, $y \neq 0$ everywhere. Let's pass to the solution of the Cauchy problem. If $y_0 = 0$ then $y \equiv 0$ is the unique possible solution. If $y_0 \neq 0$ then the solution can be only of the form $y(t) = -\frac{1}{t+C}$. By imposing the passage condition

$$y(t_0) = y_0, \iff -\frac{1}{t_0 + C} = y_0, \iff C = -t_0 - \frac{1}{y_0}.$$

There's a unique possible C hence a unique possible solution given by

$$y(t) = -\frac{1}{t - t_0 - \frac{1}{y_0}}.$$

Notice that this, as function, would be defined in $\mathbb{R}\backslash\{t_0 + \frac{1}{y_0}\} =]-\infty, t_0 + \frac{1}{y_0}[\cup]t_0 + \frac{1}{y_0}, +\infty[$. However, only one among these two intervals contains t_0. As $y_0 > 0$ it is $]-\infty, t_0 + \frac{1}{y_0}[$, while as $y_0 < 0$ it is $]t_0 + \frac{1}{y_0}, +\infty[$ (because $t_0 + \frac{1}{y_0} < t_0$). □

This example shows some interesting facts. First, the *life interval* of the solution depends by the initial condition (t_0, y_0): the interval might be bounded or unbounded, even the entire $]-\infty, +\infty[$. However, unless the solution is known, it is impossible to say *a priori* what is the case. These are general features, caught by the

Theorem 7.1.5: Local Cauchy–Lipschitz

Let $f : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathscr{C}(D)$;

ii) $\partial_y f \in \mathscr{C}(D)$.

Then, we have

- **Existence:** for every $(t_0, y_0) \in D$ there exists a solution $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ of $CP(t_0, y_0)$ fulfilling the following property: there is not a solution $\widehat{y} : I \longrightarrow \mathbb{R}$ of $CP(t_0, y_0)$ defined on $I \supsetneq]\alpha, \beta[$. Solution y is called **maximal solution**.
- **Uniqueness:** if $y : I \longrightarrow \mathbb{R}$ and $\widehat{y} : J \longrightarrow \mathbb{R}$ are two solutions of $CP(t_0, y_0)$ then $y = \widehat{y}$ on $I \cap J$.

In general, α, β and, depend by (t_0, y_0). The interval $]\alpha, \beta[$ is, by definition, the *life interval* for the maximal solution, α is called **birth time** and β is called **death time** of the solution.

Example 7.1.6. *Show that the equation*

$$y' = \frac{e^{ty}}{1+y^2}$$

fulfills the local Cauchy–Lipschitz condition on $\mathbb{R}\times\mathbb{R}$ *but* **not** *the global condition on any strip* $[a,b]\times\mathbb{R}$.

Sol. — Here $f(t,y) = \frac{e^{ty}}{1+y^2}$ is defined on $D = \mathbb{R}\times\mathbb{R}$. Clearly f is continuous and

$$\partial_y f(t,y) = \frac{te^{ty}(1+y^2) - 2ye^{ty}}{(1+y^2)^2} = e^{ty}\frac{t(1+y^2) - 2y}{(1+y^2)^2} \in \mathscr{C}(\mathbb{R}^2).$$

This shows that the local Cauchy–Lipschitz condition holds. However, $\partial_y f(t,y)$ is clearly unbounded on any strip $[a,b]\times\mathbb{R}$. Indeed: if $t\neq 0$ is fixed, $|\partial_y f(t,y)| \longrightarrow +\infty$ as $y\longrightarrow +\infty$. □

In general, the explicit determination of the life time interval for a maximal solution is difficult if not impossible. We might expect that, when the solution "expires" (in the past, as $t\longrightarrow \alpha+$. in the future, as $t\longrightarrow \beta-$) something "dramatic" should happens. Indeed, let us imagine that the "final" point $(\beta, y(\beta-))\in D$. Then, applying local existence and uniqueness, the $CP(\beta, y(\beta-))$ would have a solution $\widehat{y} :]\widehat{\alpha},\widehat{\beta}[\longrightarrow\mathbb{R}$, with $\beta\in]\widehat{\alpha},\widehat{\beta}[$. In particular, we could extend y some time after β by posing

$$\widetilde{y}(t) := \begin{cases} y(t), & t<\beta, \\ \\ \widehat{y}(t), & \beta\leqslant t<\widehat{\beta}. \end{cases}$$

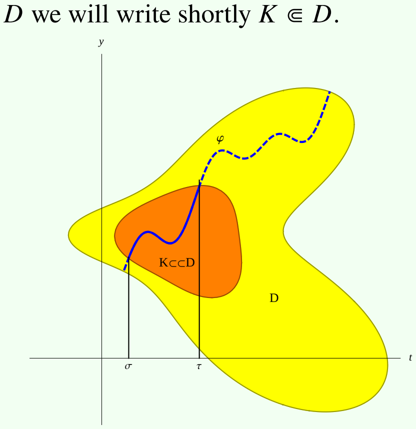
But then, since $\widetilde{y}(t_0) = y(t_0) = y_0$, $\widetilde{y}$ would solve $CP(t_0,y_0)$ and it would be defined on $]\alpha,\widehat{\beta}[\supsetneq]\alpha,\beta[$, contradicting the characteristic property of a maximal solution. The conclusion would be then that $(\beta, y(\beta-))\notin D$, or, in other words, *when a maximal solution expires it must leave the domain D*. This argument is not a proof, because there are many delicate points that should be clarified. However, it suggest a general property that turns out to be true by fixing the argument shown here. This is a fundamental property of maximal solutions:

Theorem 7.1.7: exit from compact sets

Let $f : D\subset\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be fulfilling Local CL Thm on D open. Then, every maximal solution y must leave every compact set $K\subset D$. Precisely:

$$\forall K\subset D,\ K \text{ compact } : \ \exists\sigma,\tau \ : \ (t,y(t))\notin K,\ \forall t<\sigma,\ \forall t>\tau.$$

To write K compact included in D we will write shortly $K\Subset D$.



7.2. Qualitative Study of Scalar Equations

The main goal of qualitative study is to plot the diagram of a solution of a given CP. Of course, if we have explicitly computed the solution this is a Calculus 1 exercise. But what if we cannot explicitly solve a CP? As we will see in some examples here below, the differential equation can provide the information we need to have a qualitative idea, including a diagram, of the solution. Instead of giving recipes, we prefer to illustrate the techniques through few key examples. The reader will notice that we will make an important use of the general results presented in the previous section.

Example 7.2.1. *Consider the Cauchy problem*

$$\begin{cases} y' = \dfrac{\tan y}{1+y^2}, \\ \\ y(0) = y_0, \end{cases}$$

Show that local existence and uniqueness hold. Find constant solutions and regions of D where the solutions are increasing/decreasing. Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ the maximal solution with $y_0 \in]0, \frac{\pi}{2}[$. Show that y is monotone and deduce that $\alpha = -\infty$, computing also $y(-\infty)$. Show that $\beta < +\infty$ and compute $y(\beta-)$. Show that $y \in \mathscr{C}^2$ and find the concavity of y. Use this to show again that $\beta < +\infty$ and to deduce an estimate of β. With the previous informations plot a graph of y.

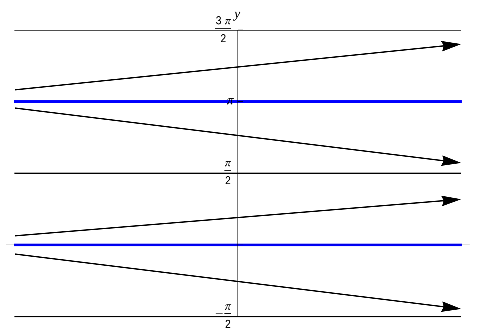
Sol. — Let $f(t,y) = \frac{\tan y}{1+y^2}$ is defined on $D :=]-\infty, +\infty[\times \left(\mathbb{R} \backslash \left\{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\right\}\right)$. On D, $f, \partial_y f$ are clearly continuous: therefore, local existence and uniqueness are fulfilled. Let $y \equiv C$ be a constant solution. To be a solution we need that

$$0 = y' = \frac{\tan y}{1+y^2} = \frac{\tan C}{1+C^2}, \iff \tan C = 0, \iff C = k\pi, \ k \in \mathbb{Z}.$$

Let y be a solution. Then

$$y \nearrow, \text{ on } I \iff y' = \frac{\tan y}{1+y^2} \geqslant 0, \text{ on } I, \iff k\pi \leqslant y < k\pi + \frac{\pi}{2}, \ k \in \mathbb{Z}.$$

By this we deduce the picture of plane regions where solutions are increasing/decreasing.



Let now $y :]a, b[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0, y_0)$ with $y_0 \in]0, \frac{\pi}{2}[$. We want to prove that

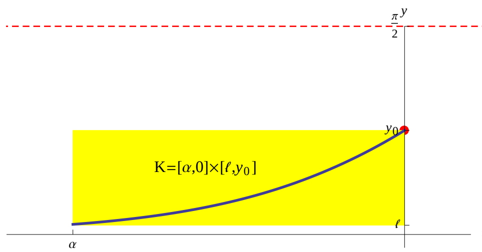
$$0 < y(t) < \frac{\pi}{2}, \ \forall t \in]a, b[, \implies \varphi \nearrow.$$

If this is true then $y \nearrow$. Suppose, by contradiction, that there exists $\exists t_1 \in]a, b[$ such that $y(t_1) \leqslant 0$. Then, by intermediate values thm, $y(\widehat{t}) = 0$ for some $\widehat{t}$. But then y must intersect a costant solution at $t = \widehat{t}$. By uniqueness, therefore, $y \equiv 0$, but this is impossible being $y(0) = y_0 > 0$. Similarly, it is impossible that there exists t_1 such that $y(t_1) \geqslant \frac{\pi}{2}$ otherwise there should be $\widehat{t}$ such that $y(\widehat{t}) = \frac{\pi}{2}$. In this case $(\widehat{t}, y(\widehat{t})) \notin D$ and this is impossible for a solution.

We know now that $y \nearrow$. Therefore, by properties of monotone functions, $\exists \lim_{t \to a+} y(t) = \ell$. Moreover, because $0 < y(t) < \frac{\pi}{2}$, we have $0 \leqslant \ell \leqslant \frac{\pi}{2}$ (actually $< \frac{\pi}{2}$). Suppose then that $\alpha > -\infty$. We should have then

$$(t, y(t)) \in [\alpha, 0] \times [\ell, y_0] =: K, \ \forall t < 0,$$

so that the solution wouldn't get out (in the past) by the compact K, and this contradicts the fugue by compacts.



The unique possibility is therefore $\alpha = -\infty$. About ℓ we have two alternatives: either $\ell = 0$ or $\ell > 0$. The second is impossible: taking the equation and passing to the limit as $t \longrightarrow -\infty$ we would have

$$y' \longrightarrow \frac{\tan \ell}{1 + \ell^2}.$$

But being $y \longrightarrow \ell$ as $t \longrightarrow -\infty$ **it cannot be** $\frac{\tan \ell}{1+\ell^2} \neq 0$. We can motivate this by a little general fact consequence of Hôpital rules:

Proposition 7.2.2

Let $\varphi \in \mathscr{C}^1$ be such that

$$\lim_{t \to \pm\infty} \varphi(t) = \ell, \quad \lim_{t \to \pm\infty} \varphi'(t) = \ell', \quad \text{with } \ell, \ell' \in \mathbb{R}^d, \quad \Longrightarrow \quad \ell' = 0.$$

Proof. Indeed

$$0 = \lim_{t \to \pm\infty} \frac{\varphi(t)}{t} \overset{(H)}{=} \lim_{t \to \pm\infty} \varphi'(t) = \ell'. \quad \square$$

Therefore $\frac{\tan \ell}{1+\ell^2} = 0$, iff $\tan \ell = 0$, iff $\ell = k\pi$, and being $0 \leqslant \ell < y_0 < \frac{\pi}{2}$ necessarily $\ell = 0$.

Let's discuss now what happens at β. We know that $y \nearrow$ so $\exists \lim_{t \to \beta-} y(t) =: \ell$. Because $0 < y(t) < \frac{\pi}{2}$ forcely $0 \leqslant \ell \leqslant \frac{\pi}{2}$ (actually $\ell \geqslant y_0$). Let's deduce by this that $\beta < +\infty$. If $\beta = +\infty$ we would have

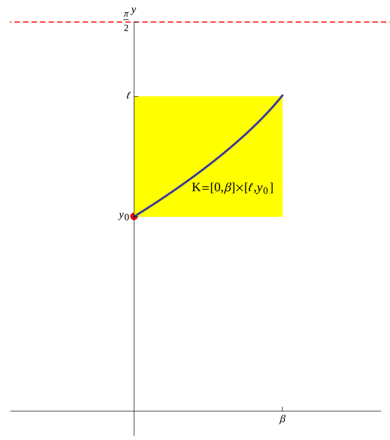
$$y'(t) \longrightarrow \frac{\tan \ell}{1 + \ell^2} =: \ell'.$$

Now: either $\ell < \frac{\pi}{2}$ or $\ell = \frac{\pi}{2}$. In the first case $\ell' \in \mathbb{R}$ but then, by previous proposition, $\ell' = 0$, that is $\frac{\tan \ell}{1+\ell^2} = 0$, iff $\ell = k\pi$. But this is impossible because $y_0 < \ell \leqslant \frac{\pi}{2}$. Therefore $\ell = \frac{\pi}{2}$, hence $\ell' = +\infty$. But also this is impossible: if $\ell' = +\infty$ then $\varphi' \geqslant 1$ definitively, therefore $y(t) \longrightarrow +\infty$ so it cannot be $y(t) \longrightarrow \frac{\pi}{2}$.

Moral: $\beta < +\infty$. Let's see that $\ell = \frac{\pi}{2}$. If $\ell < \frac{\pi}{2}$ we would have that

$$(t, y(t)) \in [0, \beta] \times [y_0, \ell] \Subset D, \ \forall t > 0,$$

contradicting the fugue by compacts.



Let's pass now to the discussion of concavity. Because $y' = \frac{\tan y}{1+y^2}$, by deriving this respect to t (do not forget that $y = y(t)$) we get

$$\begin{aligned} y'' &= \frac{(\tan y)'(1+y^2) - (\tan y)(1+y^2)'}{(1+y^2)^2} = \frac{(1+(\tan y)^2)y'(1+y^2) - (\tan y)2yy'}{(1+y^2)^2} \\ &= y'\frac{1+(\tan y)^2 + y^2 + y^2(\tan y)^2 - 2y\tan y}{(1+y^2)^2} = y'\frac{1+y^2(\tan y)^2 + (y-\tan y)^2}{(1+y^2)^2}. \end{aligned}$$

By this it is evident that $y'' \geqslant 0$ iff $y' \geqslant 0$, and because this is true by previous discussion, y is convex. Taking the tangent at $t = 0$ of equation

$$y = y_0 + \varphi'(0)t = y_0 + \frac{\tan y_0}{1+y_0^2}t,$$

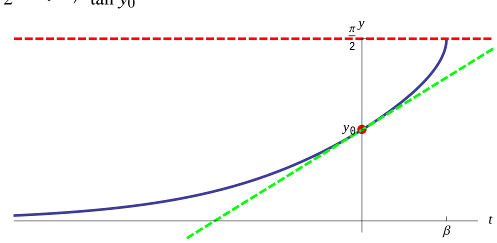
by properties of convex functions it follows that

$$y(t) \geqslant y_0 + \frac{\tan y_0}{1+y_0^2}t.$$

And because

$$y_0 + \frac{\tan y_0}{1+y_0^2}t \leqslant \frac{\pi}{2}, \quad \Longleftrightarrow \quad t \leqslant \left(\frac{\pi}{2} - y_0\right)\frac{1+y_0^2}{\tan y_0}$$

we deduce necessarily that $\beta < \left(\frac{\pi}{2} - y_0\right)\frac{1+y_0^2}{\tan y_0}$. $\square$



Example 7.2.3. *Consider the differential equation*

$$y' = \frac{1}{t^2+y^2-1}.$$

Determine the domain of local existence and uniqueness. Let now y be the solution of the CP(0,0). Show that y is odd increasing and find its concavity. With these informations estimate the lifetime of y.

Sol. — Let $f(t,y) = \frac{1}{t^2+y^2-1}$. Clearly $f \in \mathscr{C}(D)$ where $D = \{(t,y) \in \mathbb{R}^2 \ : \ t^2+y^2 \neq 1\}$ (the plane minus the circle centered in $(0,0)$ with radius 1. Moreover

$$\partial_y f = -\frac{2y}{(t^2+y^2-1)^2} \in \mathscr{C}(D).$$

Let now $y :]\alpha,\beta[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0,0)$. To show that y is odd we need to prove that

$$y(-t) = -y(t),\ \forall t \iff y(t) = -y(-t),\ \forall t$$

We will use the following standard argument: let z be defined as $z(t) := -y(-t)$. Notice that z is still a solution of the same Cauchy problem solved by y. Indeed: $z(0) = -y(-0) = -y(0) = -0 = 0$ and

$$z'(t) = (-y(-t))' = y'(-t) = \frac{1}{(-t)^2 + (y(-t))^2 - 1} = \frac{1}{t^2 + z(t)^2 - 1},$$

By uniqueness $z = y$, that is $-y(-t) = y(t)$ for every t.

Let's pass to the monotonicity. We have

$$y \searrow, \iff y' \leqslant 0,\ \forall t, \iff t^2 + y^2 < 1,\ \forall t.$$

Now, as $t = 0$ this is true (because $y(0) = 0$). If for some t it would be $t^2 + y^2 \geqslant 1$ then, either $t^2 + y(t)^2 = 1$ (but this means $(t, y(t)) \notin D$, impossible) or $t^2 + y^2 > 1$. By continuity, then, there should exists another time s where $s^2 + y^2 = 1$, impossible. Therefore $t^2 + y^2 < 1$ always.

For the concavity let's compute y''. Deriving the equation

$$y'' = -\frac{\left(t^2+y^2-1\right)'}{\left(t^2+y^2-1\right)^2} = -\frac{2t+2yy'}{(t^2+y^2-1)^2}$$

Therefore

$$y'' \geqslant 0, \iff t + yy' \leqslant 0.$$

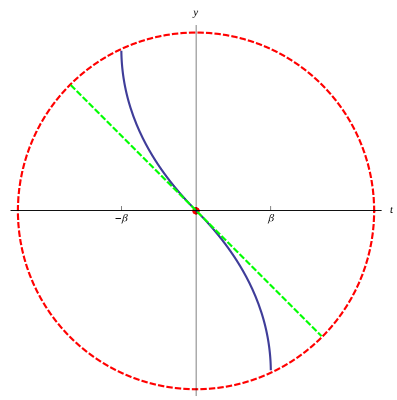
We know that $y' < 0$ always. Being $y(0) = 0$ we deduce that

$$y \geqslant 0, \iff t \leqslant 0.$$

Therefore: as $t \leqslant 0$, $y \geqslant 0$, $y' < 0$ hence $t + yy' \leqslant 0$; as $t \geqslant 0$, $y \leqslant 0$, $y' < 0$ hence $t + yy' \geqslant 0$. In conclusion: y is convex as $t \leqslant 0$, concave as $t \geqslant 0$. In particular y is below of its tangent for $t = 0$,

$$y(t) \leqslant y(0) + y'(0)t = -t,$$

and the straight line $y = -t$ crosses $t^2 + y^2 = 1$ at $t = \frac{\sqrt{2}}{2}$. It follows that $\beta < \frac{\sqrt{2}}{2}$. □



7.3. Exercises

EXERCISE 7.3.1. Consider the equation

$$y' = y\,(e^y - 1)\,.$$

Check that local existence and uniqueness hypotheses are fulfilled. Find stationaries solutions and draw in the plane (t, y) regions where solutions are increasing/decreasing. Since now, let y be the maximal solution with $y(0) = y_0 > 0$. Discuss the monotonicity and concavity of y. Find the maximal interval and the limits at the extremes of it. Plot a graph of y.

EXERCISE 7.3.2. Consider the equation

$$y' = \frac{\log y}{1 + y}.$$

Find the maximum domain where local existence and uniqueness hypotheses are fulfilled. Find stationaries solutions and region where solutions are increasing/decreasing. Since now let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution with $y(0) = y_0 \in]0, 1[$. Show that φ is monotone and find its concavity. Show that $\alpha = -\infty$ and compute $\lim_{t\to-\infty} y(t)$. Say if $\beta < +\infty$ and compute $\lim_{t\to\beta-} y(t)$ and $\lim_{t\to\beta-} y'(t)$. Plot a graph of the solution.

EXERCISE 7.3.3. Consider the Cauchy problem

$$\begin{cases} y' = t(y^2 - 1) \arctan y, \\ \\ y(0) = y_0. \end{cases}$$

Check that local existence and uniqueness holds. Find stationaries solutions and regions in the plane (t, y) where solutions are increasing/decreasing. Since now, let y be the maximal solution with $y(0) = y_0 \in]0, 1[$. Show that y is even and discuss the nature of $t = 0$ for y. Find the maximal interval, the limits at its extremes and plot a graph of the solution.

EXERCISE 7.3.4. Let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of

$$\begin{cases} y' = \dfrac{1}{\log y + t}, \\ \\ y(1) = 1. \end{cases}$$

We accept that it exists. Show that y is increasing, find its concavity, show that $\beta = +\infty$ and compute $y(+\infty)$. Show that $\alpha > -\infty$ and compute $\lim_{t\to\alpha+} y'(t)$. Plot a graph of the solution.

EXERCISE 7.3.5. Let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of

$$\begin{cases} y' = \dfrac{1}{t - \log y}, \\ \\ y(0) = e. \end{cases}$$

We accept that it exists. Of course, the graph of φ is contained into $D_> := \{(t, y) \in \mathbb{R}\times\mathbb{R} \ : \ y > e^t\}$: why? Deduce that y is increasing. Study the concavity of y. Show that if $\alpha > -\infty$ then $y(\alpha+) = +\infty$: deduce that $\alpha = \ldots$. Show that $\beta < +\infty$, and compute $\lim_{t\to\beta-} y'(t)$. Plot a graph of the solution. Show that $y(-\infty) = +\infty$.

EXERCISE 7.3.6. Consider the differential equation

$$y' = \frac{t}{y^2 - t^2}.$$

Find the maximum domain $D \subset \mathbb{R} \times \mathbb{R}$ on which local existence and uniqueness are fulfilled. Since now, let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be a maximal solution of the Cauchy problem with $y(0) = 1$. Show that: y is even, $y \nearrow$ on $[0, \beta[$ and deduce the nature of $t = 0$ for y. Show that $y(\beta-) = +\infty$. Plot a graph of the solution.

EXERCISE 7.3.7. Consider the differential equation

$$y' = \frac{\sin(ty)}{1 + y^2}.$$

Find the maximum domain $D \subset \mathbb{R} \times \mathbb{R}$ on which local existence and uniqueness are fulfilled. Do global existence and uniqueness holds? Find stationaries solutions and regions of the plane (t, y) where solutions are increasing/decreasing. Since now, let $y : \mathbb{R} \longrightarrow \mathbb{R}$ be the solution of $CP(0, 1)$. show that $\varphi > 0$ everywhere and that y is even. Compute $y''(0)$ and deduce the nature of $t = 0$ for y. Show that necessarily there exists $\tau > 0$ such that y has a maximum at τ. Compute $y(\pm\infty)$. Plot a graph of the solution.

CHAPTER 8

Systems of Differential Equations

In this Chapter, we continue our journey into ODEs extending the discussion to the case of *systems of differential equations*. For simplicity, we will limit our discussion to 2×2 *autonomous systems*, that is to systems of the form

$$\begin{cases} x' = f(x, y), \\ \\ y' = g(x, y), \end{cases} \quad \text{where } f, g : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

Systems of ODEs are used to describe the evolution of several entities interacting according to some rule. We will present some remarkable example in the next Section.

As for the previous Chapter, the main focus here is on qualitative methods for studying solutions. Differently from the case of scalar equations, however, the main focus is not on plotting the diagram of the solutions. There are two main reasons for that. The first is a technical difficulty. To do the kind of analysis we did for scalar equations, that is, draw information on x (for instance) from its equation $x' = f(x, y)$ is hard. Indeed, to get any info on x we would need to know also y. This is itself depending on x, so we end into a vicious cycle. The second is actually an opportunity. As we will see from some of the examples, in the case of a system other figures different of single solution diagrams provides more insight on the behaviour of the system. The techniques developed here can be applied to certain second order equations that we find in Physics (Newton equations).

Chapter requirements: basic and advanced differential equations,

8.1. Some Examples

In this Section we illustrate, through few examples, how systems of differential equations arise naturally from real world problems.

8.1.1. A model for a pandemic. We consider a population subject to a pandemic spread. The goal is to describe the evolution of the pandemic describing the number of individuals that have been affected. We divide the population into three groups: *susceptible* individuals, namely individuals that have not yet been affected by the virus; *infected* individuals, and immunized individuals, namely individuals that have recovered from the disease or who deceased. We denote these groups by $x(t), y(t), z(t)$ in function of time t respectively.

Let us describe the relations among these quantities, starting with $x = x(t)$. We look at a short time variation, that is $dx := x(t + dt) - x(t)$. Thus, $\frac{dx}{dt}$ represents the variation of x per unit of time. A reasonable assumption is *the variation per unit of time of susceptible individuals is negative and proportional to both the number of susceptible and of infected individuals*. Formally, this means

$$\frac{dx}{dt} = -R_0 xy,$$

where $R_0 > 0$ is the *rate of contagiousness* (the famous R_0 we heard in the news for COVID19: $R_0 = 1$ means every positive affects one susceptible), $R_0 > 1$ means that every positive affect more than 1 susceptible individual.

Actually, previous identity becomes a true identity when $dt \longrightarrow 0$, and since $\frac{x(t+dt)-x(t)}{dt} \longrightarrow x'(t)$ we get finally

$$x'(t) = -R_0 x(t)y(t).$$

On y, two phenomena determine its variation per unit of time. First, $\frac{dy}{dt}$ increases of a quote equal to the quote of susceptible individuals that become affected, that is $+R_0 x(t)y(t)$. Second, $\frac{dy}{dt}$ decreases of a quote proportional to the number of infected individuals because of recoveries or deaths.

$$\frac{dy}{dt} = +R_0 xy - \mu y,$$

or, more precisely

$$y'(t) = R_0 x(t)y(t) - \mu y(t).$$

The third quantity, number of immunized individuals, is simply driven by the equation

$$\frac{dz}{dt} = +\mu y(t), \iff z'(t) = +\mu y(t).$$

As you can see, the first two entities x and y verify a closed system of two ODEs in two unknowns:

$$\begin{cases} x'(t) = -R_0 x(t)y(t), \\ \\ y'(t) = R_0 x(t)y(t) - \mu y(t), \end{cases}$$

Once x, y is found we can determine z integrating $z' = \mu y$. Specific values for the coefficients are estimated by real observations. The difficulty with this type of systems is that apparently we have to solve two equations "simultaneously".

8.1.2. Prey–predator system. A second example we consider and ecosystem made of two species: *predators* and *preys*. You may imagine predators = lions and preys = gazelles, for instance. Predators kill preys because these are their food, while preys food is in the natural environment. Food determines how each population varies. For predators, their growth (variation of population per unit of time) is proportional to both populations. However, in absence of preys, predators naturally die. For preys, their growth is proportional to the population and, at same time, they die proportionally to their number and number of predators.

Formalizing these principles, let $x = x(t)$ the number of preys at time t and $y = y(t)$ the number of predators at same time. We represent with $dx = x(t+dt) - x(t)$ the variation of preys on the time-frame $[t, t+dt]$, $\frac{dx}{dt}$ the variation per unit of time. We have

$$\frac{dx}{dt} \approx \underbrace{-\lambda xy}_{predators\ effect} + \underbrace{\mu y}_{natural\ growth},$$

Here, λ, μ are positive constants, specific for each specie of preys and predators. Previous relation becomes exact when $dt \longrightarrow 0$, this leading to the equation

$$x' = -\lambda xy + \mu x.$$

Similarly, for predators we have

$$\frac{dy}{dt} \approx \underbrace{+\nu xy}_{growth} - \underbrace{\kappa x}_{natural\ deceases}.$$

where, also in this case, ν, κ are positive constant characteristic for each prey-predator species. Previous relation becomes a true identity in the limit $dt \longrightarrow 0$,

$$y' = +\nu xy - \kappa y.$$

Coupling the two equations we obtain the so called *prey-predator system*:

$$\begin{cases} x' = -\lambda xy + \mu x, \\ \\ y' = +\nu xy - \kappa y. \end{cases}$$

8.2. Cauchy Problem: Existence and Uniqueness

Let

$$f = f(x,y),\ g = g(x,y)\ :\ \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

Cauchy Problem for a 2×2 system can be naturally stated as

$$CP(t_0,(x_0,y_0))\ :\ \begin{cases} x' = f(x,y), \\ y' = g(x,y), \\ x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases} \tag{8.2.1}$$

A solution is now a couple $x, y \in \mathscr{C}^1(I)$ fulfilling the system. In particular, $(x(t), y(t)) \in \Omega$ for all $t \in I$, hence also $(x_0, y_0) \in \Omega$. The fact that the system is *autonomous*, that is f, g do not depend by t explicitly, has a nice remarkable consequence on solutions:

Proposition 8.2.1

$(x(t), y(t))$ solves $CP(t_0,(x_0,y_0))$ iff $(x(t-t_0), y(t-t_0))$ solves $CP(0,(x_0,y_0))$.

In particular, for the Cauchy Problem we can conventionally consider $t_0 = 0$. Existence and uniqueness for CP (8.2.1) follows similar results as for the scalar case (be careful: here f, g are not depending by t explicitly). For instance we have

Theorem 8.2.2: Global Cauchy–Lipschitz

Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathscr{C}(\mathbb{R}^2)$;

ii) $\partial_x f,\ \partial_y f,\ \partial_x g,\ \partial_y g$ are bounded in $\mathbb{R}^2$, that is

$$\exists L > 0\ :\ |\partial_x f(x,y)|,\ |\partial_y f(x,y)|,\ |\partial_x g(x,y)|,\ |\partial_y g(x,y)| \leqslant L,\ \forall (x,y) \in \mathbb{R}^2.$$

Then, for every $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique solution to $CP(0,(x_0,y_0))$.

This Theorem suffers the same critics of its homologous for scalar equations. A considerably weaker version is the

Theorem 8.2.3: Local Cauchy–Lipschitz

Let $f, g : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathscr{C}(\Omega)$;

ii) $\partial_x f,\ \partial_y f,\ \partial_x g,\ \partial_y g \in \mathscr{C}(\Omega)$.

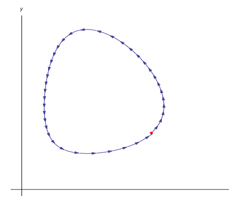
Then, for every $(x_0, y_0) \in \Omega$ there exists a unique maximal solution to $CP(0,(x_0,y_0))$. Such maximal solution is defined on a life time interval $]\alpha, \beta[$ with the following property: it is not possible to extend (x, y) to $J \supsetneq]\alpha, \beta[$ as solution.

8.3. Orbits and Phase Portrait

Since now, we will assume that Local CL Thm Hypotheses are fulfilled. Let's consider a maximal solution (x, y). To have an idea on its behavior, we could try to plot the diagrams of $x = x(t)$ and $y = y(t)$. This, however, might not be the most incisive way to describe the qualitative behavior for a system. Indeed, the most interesting qualitative information concerns the couple (x, y) not just the single x and y. The natural graph would be that of the function

$$t \longmapsto (x(t), y(t)),$$

but this is hard to be described being that of a curve in the space t, x, y. A good compromise is the following: we may see point $(x(t), y(t))$ as a point in movement in the plane xy, the *state space* of the system.



The curve $t \longmapsto (x(t), y(t))$ represents the evolution of the state of the system, so in some way its "trace" on the state space reveals a qualitative behavior. For instance, if the point $(x(t), y(t))$ is moving from left to right it means that $x \nearrow$. Let's introduce the important

Definition 8.3.1

Given a maximal solution (x, y) defined on $]\alpha, \beta[$, we call **orbit** of the solution the set

$$\gamma(x, y) := \{(x(t), y(t)) \;:\; t \in]\alpha, \beta[\} \subset \Omega.$$

Among all the possible solutions, constant solutions are very important for applications: they represent states where nothing happens.

Definition 8.3.2

A point $(x_0, y_0) \in \Omega$ is said **equilibrium** if $(x, y) \equiv (x_0, y_0)$ is a constant solution.

Remark 8.3.3.

$$(x_0, y_0) \text{ equilibrium} \iff \begin{cases} f(x_0, y_0) = 0, \\ \\ g(x_0, y_0) = 0. \end{cases}$$

So, for instance, the orbit of a constant solution reduces to the singleton $\{(x_0, y_0)\}$, where (x_0, y_0) is an equilibrium. In general, however, $\gamma(x, y)$ is a curve in the plane. As consequence of the existence and uniqueness, a certain number of elementary properties of orbits:

- every $(x_0, y_0) \in \Omega$ belongs precisely to a unique orbit;
- if two orbits have a common point they must coincide;
- a solution is *periodic*, that is if there exists T such that $(x(t + T), y(t + T)) = (x(t), y(t))$ for every t, if and only if its orbit is a closed circuit in the plane.

A less immediate property, that actually follows by a suitable extension of the argument of exit by compact sets, is the following:

Corollary 8.3.4

If an orbit $\gamma(x, y) \subset K \subset \Omega$ where K is compact (that is, closed and bounded), then the maximal interval for the solution (x, y) is $]-\infty, +\infty[$. In particular, if $\gamma(x, y)$ is a closed circuit, the solution (x, y) is periodic.

The way the point $(x(t), y(t))$ moves along $\gamma(x, y)$ is a fundamental qualitative information. Noticed that the tangent vector to $(x(t), y(t))$ is $(x'(t), y'(t))$, we can use this vector to define the **orientation** of the orbit. Qualitatively, we might distinguish four possible orientations:

- $\nearrow$, when $x \nearrow$ and $y \nearrow$;
- $\searrow$, when $x \nearrow$ and $y \searrow$;
- $\nwarrow$, when $x \searrow$ and $y \nearrow$;
- $\swarrow$, when $x \searrow$ and $y \searrow$.

Notice that

$$x \nearrow, \iff x' \geqslant 0, \iff f(x, y) \geqslant 0, \quad \text{while} \quad y \nearrow, \iff y' \geqslant 0, \iff g(x, y) \geqslant 0$$

An orbit endowed with its orientation is called **oriented orbit**. We call **phase portrait** the plot of the typical oriented orbit of the system. Our goal will be to develop methods to plot the phase portrait (or parts of it) for a given system.

8.4. First Integrals

The plot of the phase portrait for a general 2×2 system

$$\begin{cases} x' = f(x, y), \\ \\ y' = g(x, y), \end{cases} \tag{8.4.1}$$

can be very difficult. Even in the case of linear systems, as shown in the previous section, the discussion is non trivial. In this Section we will see a method to get a picture of the phase portrait in a certain number of cases.

The starting remark is the following. Given a solution $(x(t), y(t))$, $t \in I$, its orbit is the set

$$\{(x(t), y(t)) \; : \; t \in I\}.$$

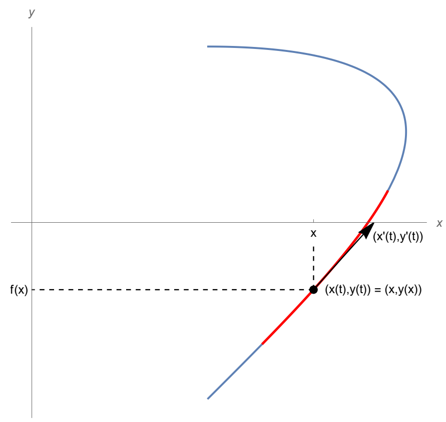
This set is, in general, the trace left by a plane curve. We may also expect that, under certain circumstances, this curve can be represented also as a diagram of type $y = y(x)$ or $x = x(y)$. Here we are a bit abusing with the notation:

- $y = y(t)$ is the $y-$solution of the system;
- $y = y(x)$ is the ordinate of the point (x, y) along the orbit having abscissa x.

So, accepting a bit of ambiguity, with this notation

$$y(t) = y(x(t)),$$

where it is clear that the two y present in this identity have different meanings. Now, fix a time t and consider the point $(x(t), y(t)) = (x, y(x))$ on the orbit.



In particular let us look at the tangent to the orbit. On one side, this is provided by the vector $(x'(t), y'(t))$ whose angular coefficient is

$$\frac{y'(t)}{x'(t)}.$$

On the other side, the same angular coefficient is $\frac{d}{dx}y(x)$. Thus

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{g(x(t), y(t))}{f(x(t), y(t))} = \frac{g(x, y)}{f(x, y)}.$$

The conclusion is that *if $y = y(x)$ is a function whose diagram is the orbit then, necessarily, $y = y(x)$ solves the differential equation*

(8.4.2) $$\boxed{\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.}$$

This equation is called **total equation**.

EXAMPLE 8.4.1. *By using the total equation, determine the orbits of the system*

$$\begin{cases} x' = y, \\ y' = -x. \end{cases}$$

SOL. — The total equation is

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This equation can be solved by separation of variables:

$$y\, dy = -x\, dx, \iff \frac{y^2}{2} = -\frac{x^2}{2} + k, \iff y^2 + x^2 = k. \quad \square$$

The previous example illustrates a situation that can be formalized in general:

Proposition 8.4.2

Let the total equation be a separable variables equation
$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \equiv \frac{a(x)}{b(y)}.$$
Then, if
$$E(x,y) := \int b(y)\,dy - \int a(x)\,dx \tag{8.4.3}$$
orbits are contained into the level sets of E, that is into sets $\{E \equiv c\}$ for some $c \in \mathbb{R}$.

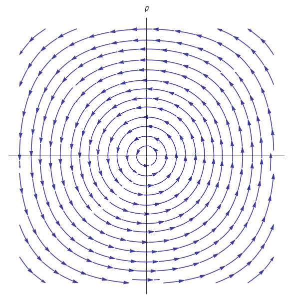
Proof. Consider a couple $(x(t), y(t))$ solution of the system. Then,
$$\frac{d}{dt}E(x(t),y(t)) = \partial_x E(x(t),y(t))x'(t) + \partial_y E(x(t),y(t))y'(t).$$
Now,
$$\partial_x E = \partial_x \left(\int b(y)\,dy - \int a(x)\,dx \right) = -a(x), \quad \partial_y E = \partial_y \left(\int b(y)\,dy - \int a(x)\,dx \right) = b(y).$$
Moreover, because of the equation $x' = f(x,y)$ and $y' = g(x,y)$. Therefore
$$\frac{d}{dt}E(x(t),y(t)) = -a(x(t))f(x(t),y(t)) + b(y(t))g(x(t),y(t)) \equiv 0,$$
because, by hypothesis $\frac{g(x,y)}{f(x,y)} \equiv \frac{a(x)}{b(y)}$, that is $a(x)f(x,y) \equiv b(y)g(x,y)$. □

Definition 8.4.3

A function $E : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called **first integral** for system (8.4.1) if
$$E(x(t),y(t)) = \text{costant}, \ \forall \text{ solution } (x(t),y(t)).$$

Example 8.4.4. *Consider the system*
$$\begin{cases} x' = -\sinh y, \\ y' = \sinh x, \end{cases}$$
i) Find the stationary points. ii) Find a non trivial first integral. iii) Plot the phase portrait of the system.

Sol. — i) $(x,y) \equiv (\xi,\eta)$ is a stationary point iff
$$\begin{cases} 0 = -\sinh \eta, \\ 0 = \sinh \xi, \end{cases} \iff (\xi,\eta) = (0,0).$$
ii) Let's write the total equation:
$$\frac{dy}{dx} = -\frac{\sinh x}{\sinh y}, \iff \sinh y\,dy = -\sinh x\,dx, \iff \cosh y = -\cosh x + c.$$
Notice that the surfaces $\{E = E_0\}$ are clearly closed (being E continuous) and bounded (because it is easy to check that $\lim_{(x,y)\to\infty_2} E(x,y) = -\infty$).



It is also easy to find out the orientation of the orbits: by the system we have that

- $x' > 0$, iff $-\sinh y > 0$, iff $y < 0$: therefore $x \nearrow$ when $y < 0$;
- $y' > 0$, iff $\sinh x > 0$, iff $x > 0$: therefore $y \nearrow$ when $x > 0$. □

Notice that a *trivial* first integral always exists: just take any constant function E. Of course, this kind of integral is completely useless: the level sets $\{E = c\}$ are empty or $\mathbb{R}^2$ according to the value of c. Moreover, the first integral doesn't contain any information on the orientation of the orbits, as the following example shows.

Example 8.4.5.

$$\begin{cases} x' = y, \\ \\ y' = -x, \end{cases} \quad e \quad \begin{cases} x' = -y, \\ \\ y' = x, \end{cases}$$

Sol. — In both cases the total equation is the same:

$$\frac{dy}{dx} = \frac{x}{-y} = -\frac{x}{y}, \quad \frac{dy}{dx} = \frac{x}{-y} = -\frac{x}{y},$$

and leads easily to the first integral $E(x, y) = x^2 + y^2$. □

An important remark is that *in general a level set $\{E = c\}$ do not coincide with a single orbit, but it may contains several orbits.*

Example 8.4.6. *Find stationary points, non trivial first integral and phase portrait of the system*

$$\begin{cases} x' = y(x - y), \\ \\ y' = -x(x - y), \end{cases}$$

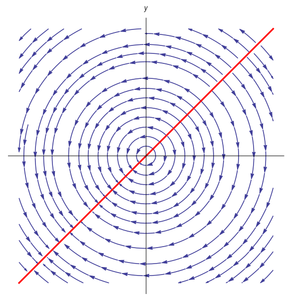
Sol. — $(x, y) \equiv (a, b)$ is stationary point iff

$$\begin{cases} 0 = b(a - b), \\ \\ 0 = -a(a - b). \end{cases} \iff (a, b) = (0, 0), \text{ or } a = b.$$

Therefore, all points (infinitely many) of the straight line $y = x$ are stationary points. The total equation is

$$\frac{dy}{dx} = \frac{-x(x - y)}{y(x - y)} = -\frac{x}{y}, \iff y\,dy = -x\,dx, \iff E(x, y) := x^2 + y^2 = c.$$

On the level line $E(x, y) = E_0$ with $E_0 > 0$ there're two stationary points (hence two orbits) as well as other two non stationary orbits (two half circles). It is indeed evident that any of the two half circles is a separate orbit because a solution $(x(t), y(t))$ cannot jump from one half circle to the other being a continuous function. It remains therefore all time in one of the two half. The orientation is easily deduced. □



Example 8.4.7 (prey–predator system). *Find the phase portrait in the first quarter of the system*

$$\begin{cases} x' = -\lambda xy + \mu x, \\ y' = +\nu xy - \kappa y. \end{cases}$$

Sol. — Easily local existence and uniqueness conditions are fulfilled but nor the global ones. We look for a first integral. The total equation is

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = \frac{\nu xy - \kappa y}{-\lambda xy + \mu x} = \frac{y(\nu x - \kappa)}{x(-\lambda y + \mu)}, \iff \left(\frac{\mu}{y} - \lambda\right) dy = \left(\nu - \frac{\kappa}{x}\right) dx$$

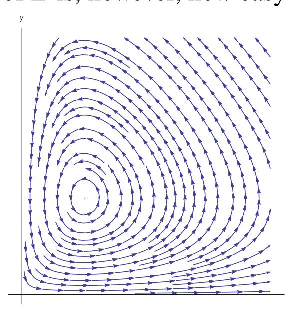
Integrating the equation we obtain

$$\mu \log|y| - \lambda y = \int \left(\frac{\mu}{y} - \lambda\right) dy = \int \left(\nu - \frac{\kappa}{x}\right) dx = \nu x - \lambda \log|x| + c.$$

Thus,

$$E(x,y) = \lambda \log|x| - \kappa x - \mu \log|y| + \nu y,$$

is a first integral. The plot of the level sets of E is, however, now easy: we limit here to reproduce the plot.



The picture shows clearly a cyclical behavior of the system. □

First Integrals are also useful to find explicit solutions of the systems through a *dimensional reduction*. To show this, assume that E be a non trivial first integral. Hence

$$E(x(t), y(t)) \equiv E(x(0), y(0)) =: E_0.$$

This is an algebraic equation that could be used to express one of the two functions in terms of the other, for instance

$$y(t) = \Phi(x(t), E_0). \tag{8.4.4}$$

Then, by the first equation of the system (8.4.1) we have

$$x'(t) = f(x(t), \Phi(x(t), E_0)),$$

which is an ODE involving only x. This can be studied and in some cases even solved finding the x of the couple solution. By (8.4.4) the y can be deduced.

Example 8.4.8. *Consider the system*

$$\begin{cases} x' = xy, \\ y' = -x^2 + 2x^4. \end{cases}$$

i) Find stationary points. ii) Find a non trivial first integral. iii) Plot the phase portrait of the system: are there periodic or global solutions? iv) Find the x solution of the Cauchy problem $x(0) = 2$, $y(0) = 2\sqrt{3}$.

Sol. — i) A point (a, b) is a stationary solution iff

$$\begin{cases} ab = 0, \\ 2a^4 - a^2 = 0, \end{cases} \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a^2(2a^2 - 1) = 0, \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a = \pm\frac{1}{\sqrt{2}}.$$

All points $(0, b)$ ($b \in \mathbb{R}$) and $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ are stationary points.

ii) The total equation is

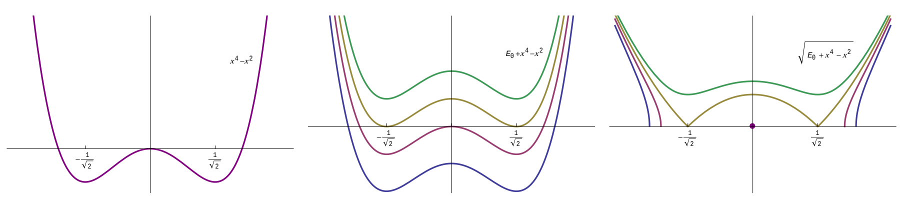
$$xy\ dy = (-x^2 + 2x^4)\ dx, \iff y\ dy = (-x + 2x^3)\ dx, \iff \frac{y^2}{2} = -\frac{x^2}{2} + \frac{x^4}{2} + C,$$

by which $E(x, y) = y^2 + x^2 - x^4$ is a non trivial first integral.

iii) We have to plot lines $E \equiv E_0$, that is

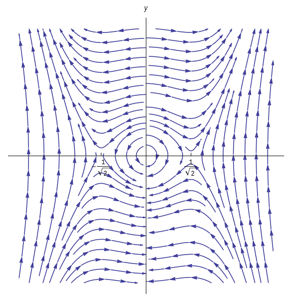
$$y^2 + x^2 - x^4 \equiv E_0, \iff y = \pm\sqrt{E_0 + x^4 - x^2}.$$

It is easy to plot $x^4 - x^2$: it is even, $(x^4 - x^2)' = 4x^3 - 2x = 2x(2x^2 - 1) = 0$ as $x = 0, \pm\frac{1}{\sqrt{2}}$. Easily we deduce the monotonicity and we see that $x = 0$ is a local max, $x = \pm\frac{1}{\sqrt{2}}$ are global mins. To plot the level lines we have to translate up and down this graph, hence take $\pm$ the root, reminding that the root follows the behavior of its argument ($\sqrt{0} = 0$, $\sqrt{\diamond}$ has a minimum/increase/decrease when $\diamond$ has a minimum/increase/decrease). The result is the following:



By this it is easy to plot the level sets $E = E_0$. Finally the orientation is

$$x \nearrow, \iff x' = xy > 0, \iff (x, y) \in \text{first and third quarter.}$$



It is evident that, apart the constant solutions, there're periodic solutions on energy levels $0 < E_0 < \frac{1}{2}$ on the component of the level sets with $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. These solutions are also global. The case $E_0 = \frac{1}{2}$ produces also two non constant non periodic globally defined solutions.

iv) Our solution fulfills $E(x, y) = E(x(0), y(0)) = E(2, 2\sqrt{3}) = (2\sqrt{3})^2 + 2^2 - 2^4 = 12 + 4 - 16 = 0$, hence it belongs to the level set $E = 0$. Then $y^2 = x^4 - x^2$. Because $x(0), y(0) > 0$, at least in a neighborhood of $t = 0$ we must have $y = \sqrt{x^4 - x^2} = x\sqrt{x^2 - 1}$. Replacing in the first equation we obtain

$$x' = xy = x \cdot x\sqrt{x^2 - 1} = x^2\sqrt{x^2 - 1}, \iff \frac{x'}{x^2\sqrt{x^2 - 1}} = 1.$$

This is a separable variable equation: integrating and setting $u = x(t)$

$$t + C = \int \frac{1}{u^2\sqrt{u^2 - 1}}\, du = \int \frac{1}{u^2\sqrt{u^2 - 1}}\, du \overset{u = \cosh v,\ du = \sinh v\ dv}{=} \int \frac{1}{(\cosh v)^2\sqrt{(\cosh v)^2 - 1}} \sinh v\ dv.$$

Now $\sqrt{(\cosh v)^2 - 1} = \sqrt{(\sinh v)^2} = |\sinh v| = \sinh v$ if $v \geqslant 0$, hence

$$t + C = \int \frac{1}{(\cosh v)^2}\, dv = \tanh v, \iff v = \tanh^{-1}(t + C), \iff x(t) = \cosh^{-1}\left(\tanh^{-1}(t + C)\right).$$

By imposing the initial condition we find C. □

8.5. Conservative systems

The method presented in the previous Section can be applied to important second order differential equations of the form

$$y'' = \partial_y V(y), \quad \text{where } V : D \subset \mathbb{R} \longrightarrow \mathbb{R}. \tag{8.5.1}$$

This type of equation is very important because it represents the paradigm for *Newtonian conservative systems*. The reason why these equations are called *conservative* is because there's an important mechanical quantity conserved along the solutions:

Proposition 8.5.1

If y is a solution of (8.5.1) then the *mechanical energy*

$$E(y, y') := \frac{1}{2}y'^2 - V(y),$$

is constant.

Proof. This is easy to check:

$$\frac{d}{dt}E(y,y') = \frac{d}{dt}\left(\frac{1}{2}y'^2 - V(y)\right) = \frac{1}{2}2y'y'' - \partial_y V(y)y' = y'\left(y'' - \partial_y V(y)\right) \equiv 0. \quad \square$$

What is the connection with the discussion done in the previous section? We can always transform an equation like (8.5.1) into an equivalent 2×2 system: indeed, setting $q = y$, $p = y'$ (we adopt here the canonical mechanical letters), then

$$\begin{cases} q' = p =: f(q,p), \\ \\ p' = \partial_q V(q) =: g(q,p). \end{cases} \tag{8.5.2}$$

Mechanical energy turns out to be a first integral for this system. Indeed, the total equation leads to

$$\frac{dp}{dq} = \frac{\partial_q V(q)}{p}, \iff p\,dp = \partial_q V(q)\,dq, \iff \frac{p^2}{2} = V(q) + c, \iff \frac{1}{2}p^2 - V(q) = c,$$

and of course $E(q,p) = \frac{1}{2}p^2 - V(q)$ is nothing but the mechanical energy written with other letters. This correspondance makes natural to adopt the language and the methods developed in the previous section but be careful:

- an orbit is a curve in the space $(q,p) = (y,y')$ (space×velocity/momentum) which is called *phase space*;
- the orientation of orbits follows a simple rule: by the first equation of the system (8.5.2)

$$q \nearrow \iff q' = p \geqslant 0, \iff (q,p) \text{ belongs in the upper half plane of the phase space.}$$

Moreover the equivalent system is not always really needed. For instance in this case *the conservation of the energy offers a non trivial and powerful reduction of the order of the equation*. Let's see in general why: writing the conservation of the energy we have

$$\frac{1}{2}(y')^2 - V(y) = E_0. \tag{8.5.3}$$

This is a *first order equation not in normal form*. In this case it is however easy to extract y' reducing effectively the order of the (8.5.1), because

$$y' = \pm\sqrt{2(E_0 - V(y))}.$$

A slightly delicate question concern the sign. Because these are two equations, which one should be considered really? Unless you're in the delicate case when $y'(0) = 0$, if for instance $y'(0) > 0$ it is clear that at least for a neighborhood of $t = 0$ $y' > 0$ hence the correct equation must be that one with $+$.

Example 8.5.2 (pendulum without friction). *Find energy and trace of orbits in phase space for the pendulum without friction*

$$m\ell\theta''(t) = -mg\sin(\theta(t)).$$

Sol. — We assume $m = 1$. First notice that the equation may be rewritten as

$$\theta'' = -\frac{g}{\ell}\sin\theta = \frac{g}{\ell}\partial_\theta\cos\theta = \partial_\theta\left(\frac{g}{\ell}\cos\theta\right)$$

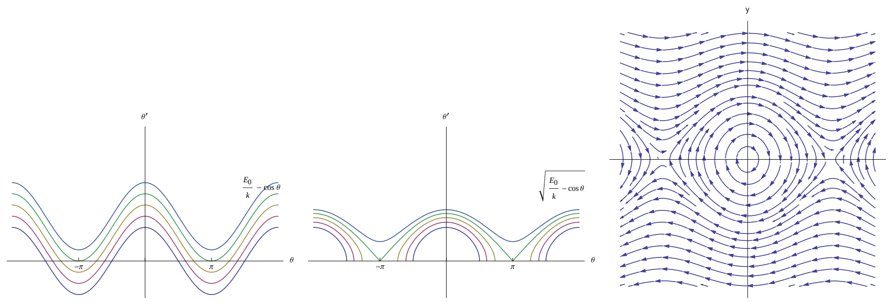
that is the equation is conservative. The mechanical energy is

$$E(\theta,\theta') = \frac{1}{2}\theta'^2 - \frac{g}{l}\cos\theta.$$

Now

$$E(q,p)=E_0, \iff \frac{\theta'^2}{2}-\frac{g}{l}\cos\theta=E_0, \iff \theta'^2=2\left(E_0+\frac{g}{l}\cos\theta\right), \ \ \theta'=\pm\sqrt{2\left(E_0+\frac{g}{l}\cos\theta\right)}.$$

It is not very difficult to plot the surfaces $E=E_0$. The orientation being standard we obtain the following picture:



By looking at the picture there're some interesting remarks. We see the classical periodic oscillatory motion (the closed cycles). The limit case when the cycle closes on an equilibrium correspond to a non periodic motion reaching in an infinite time in the future/past the equilibrium position $\theta=\pi$. Finally the not closed orbits in the upper and lower half plane corresponds to rotations: when the mass receive initially more than certain minimum energy the mass rotate infinitely many times. This is visible being θ an increasing/decreasing according to the direction anti/clockwise of the motion. □

Example 8.5.3. *Consider the differential equation*

$$y''=y^2-y, \quad (\star).$$

It is easy to check that local existence and uniqueness are fulfilled. Find stationary solutions. Show that if $\varphi: I\longrightarrow\mathbb{R}$ is a solution of $(\star)$ then also $\psi: -I\longrightarrow\mathbb{R}$, $\psi(t)=\varphi(-t)$ is a solution. Let now φ be a maximal solution of the Cauchy problem with initial conditions $\varphi(0)=a$, $\varphi'(0)=0$. Show that φ is even. Find the energy of the system and use this to find explicitly the solution of the Cauchy problem $\varphi(0)=3$, $\varphi'(0)=\sqrt{3}$.

Sol. — We have that $\varphi(t)\equiv C$ is a solution iff $0=C^2-C=C(C-1)$, that is for $C=0$, $C=1$.

Let $\varphi: I\longrightarrow\mathbb{R}$ be a solution and let $\psi: -I\longrightarrow\mathbb{R}$, be such that $\psi(t):=\varphi(-t)$. Therefore $\psi'(t)=-\varphi'(-t)$, $\psi''(t)=\varphi''(-t)$ hence

$$\psi''(t)=\varphi''(-t)=\varphi(-t)^2-\varphi(-t)=\psi(t)^2-\psi(t),$$

that is ψ is a solution.

Let $\varphi:]\alpha,\beta[\longrightarrow\mathbb{R}$. We have seen that $\psi:]-\beta,-\alpha[\longrightarrow\mathbb{R}$, $\psi(t):=\varphi(-t)$ is a solution. Moreover $\psi(0)=\varphi(0)=a$ and $\psi'(0)=-\varphi'(0)=0$. Therefore ψ solves the same Cauchy problem solved by φ: by uniqueness the two are equal where both defined. Easily it follows that $\alpha=-\beta$ (by maximality) hence $\varphi(t)=\varphi(-t)$ for every $t\in]-\beta,\beta[$.

We may see $(\star)$ as a Newton equation (mass $m=1$) with force

$$F(y)=y^2-y=\nabla_y\left(\frac{y^3}{3}-\frac{y^2}{2}\right).$$

This means that $V(y)=\frac{y^3}{3}-\frac{y^2}{2}$ is the potential so the energy in the phase space $q=y$, $p=my'=y'$ is

$$E(q,p):=\frac{p^2}{2}-\left(\frac{q^3}{3}-\frac{q^2}{2}\right).$$

Let φ then the maximal solution of the Cauhcy problem with $\varphi(0) = 3$, $\varphi'(0) = 3$. Because $E(3,\sqrt{3}) = \frac{9}{2} - \left(\frac{27}{3} - \frac{9}{2}\right) = 9 - 9 = 0$, the solution must fulfills

$$E(\varphi,\varphi') \equiv 0, \iff \frac{\varphi'^2}{2} - \frac{\varphi^2}{2}\left(\frac{2}{3}\varphi - 1\right) \equiv 0, \iff \varphi'^2 \equiv \varphi^2\left(\frac{2}{3}\varphi - 1\right), \iff \varphi' = \pm|\varphi|\sqrt{\frac{2}{3}\varphi - 1}.$$

Now $\varphi'(0) = 3$ while $\varphi(0) = 3$, so, at least in a neighborhood of initial time $t = 0$ we have

$$\varphi' = \varphi\sqrt{\frac{2}{3}\varphi - 1}, \iff \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} = 1, \iff \int \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} = t + c.$$

Now

$$\int \frac{\varphi'}{\varphi\sqrt{\frac{2}{3}\varphi - 1}} \overset{u=\varphi(t)}{=} \int \frac{1}{u\sqrt{\frac{2}{3}u - 1}}\,du \overset{v=\sqrt{\frac{2}{3}u-1},u=\frac{3}{2}(v^2+1),\ du=3\ dv}{=} \int \frac{1}{\frac{3}{2}(v^2+1)v}3v\ dv = 2\arctan v$$

$$= 2\arctan\sqrt{\frac{2}{3}u - 1} = 2\arctan\sqrt{\frac{2}{3}\varphi - 1},$$

hence

$$2\arctan\sqrt{\frac{2}{3}\varphi(t) - 1} - \frac{\pi}{2} = t + c.$$

Setting $t = 0$ we find $2\arctan 1 = c$, that is $c = \frac{\pi}{2}$, so the solution is $\varphi(t) = \frac{3}{2} + \frac{3}{3}\left(\tan\frac{2t+\pi}{4}\right)^2$. $\square$

8.6. Exercises

Exercise 8.6.1. Solve the following Cauchy problems:

$$1.\ \begin{cases} x' = -2x + \frac{y}{2}, \\ y' = 2x - 2y, \\ x(0) = 0, \\ y(0) = 1. \end{cases} \quad 2.\ \begin{cases} x' = 5x + 4y, \\ y' = x + 2y, \\ x(0) = 2, \\ y(0) = 3. \end{cases} \quad 3.\ \begin{cases} x' = x - y, \\ y' = 5x - y, \\ x(0) = 1, \\ y(0) = 1. \end{cases} \quad 4.\ \begin{cases} x' = 9x - 4y, \\ y' = 12x - 5y, \\ x(0) = 1, \\ y(0) = 0. \end{cases}$$

Exercise 8.6.2. Consider the system

$$\begin{cases} x' = y, \\ \\ y' = -x + x^3. \end{cases}$$

i) Determine the stationary solutions (if any). ii) Determine a non constant first integral for the system. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions? v) Solve the Cauchy problem $x(0) = 2$, $y(0) = 2$.

Exercise 8.6.3. Consider the system

$$\begin{cases} x' = 2y(y - 2x), \\ \\ y' = (1 - x)(y - 2x). \end{cases}$$

i) Determine the stationary solutions (if any). ii) Determine a non constant first integral for the system. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions?

Exercise 8.6.4. Consider the system

$$\begin{cases} x' = x(1+y), \\ y' = -y(1+x). \end{cases}$$

i) Determine the stationary solutions (if any). ii) Determine a non constant first integral for the system. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions? v) Solve the Cauchy problem $x(0) = -1$, $y(0) = 1$.

Exercise 8.6.5. Consider the system

$$\begin{cases} x' = 2x^2 y, \\ y' = y^2 x + x. \end{cases}$$

i) Determine the stationary solutions (if any). ii) Determine a non constant first integral for the system. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions? v) Solve the Cauchy problem $x(0) = 1$, $y(0) = 0$.

Exercise 8.6.6. Consider the equation

$$y'' = y^3 - y.$$

i) Find stationary solutions. ii) Determine a first integral. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions? v) Determine the solution of the Cauchy problem $y(0) = 2$, $y'(0) = 2$.

Exercise 8.6.7. Consider the equation

$$y'' = \sin(2y).$$

i) Find stationary solutions. ii) Determine a first integral. iii) Plot the phase portrait of the system. iv) What can be said about periodic solutions? And what about global solutions? v) Determine the solution of the Cauchy problem $y(0) = \frac{\pi}{4}$, $y'(0) = -1$.

Exercise 8.6.8. Consider the equation

$$y'' = \cosh y \sinh y.$$

i) Find stationary solutions. ii) Show that if y is a solution, then also $y(-t)$ and $-y(t)$ are solutions. For which values of $a, b \in \mathbb{R}$ is the solution of the Cauchy problem $y(0) = a$, $y'(0) = b$ even? For which a, b, is it odd? iii) Determine a first integral. iv) Plot the phase portrait of the system. What can be said about periodic solutions? And what about global solutions? v) Determine the solution of the Cauchy problem $y(0) = \log(1 + \sqrt{2})$, $y'(0) = 1$.

CHAPTER 9

Holomorphic functions

9.1. Introduction

We studied the differentiability for functions $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. In particular, we realized that the definition of a derivative is not straightforward:

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}{\vec{h}}$$

does not make sense because we cannot divide by vectors. In this Chapter we discuss again differentiability on functions $f = f(z) : D \subset \mathbb{C} \longrightarrow \mathbb{C}$. Apparently, $\mathbb{C}$ is "like" $\mathbb{R}^2$. However, limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

makes sense now, because we have a division in $\mathbb{C}$. What is astonishing is how different is the $\mathbb{C}$−differentability from $\mathbb{R}$ or $\mathbb{R}^2$ differentiability. This because of a number of non trivial exceptional properties. For instance:

- a $\mathbb{C}$−differentuable function is automatically $\mathscr{C}^\infty$ (this is completely false with $\mathbb{R}$−differentiability; for example, $f(x) = x|x|$ is differentiable at $x = 0$, but $f'(x) = 2x\mathrm{sgn}x = 2|x|$ is not);
- a bounded $\mathbb{C}$−differentiable function is necessarily constant (also this is false in $\mathbb{R}$: take $f(x) = \sin x$);
- if a $\mathbb{C}$−differentable function has zeroes (c_n) such that $c_n \longrightarrow c_\infty \in \mathbb{C}$, then $f \equiv 0$.

These and other properties show how exceptional is $\mathbb{C}$−differentiability, so that one may wonder if these functions are only a mathematical curiosity. This is not the case for certain reasons. First, most of the important elementary functions (like exp, sin, cos, sinh, cosh) can be naturally extended to $\mathbb{C}$ and they are $\mathbb{C}$−differentiable functions. Second, in many applied problems one deals with this type of functions introduced ad by product of some *integral transform* (Fourier and Laplace and many others). And third, $\mathbb{C}$−differential calculus provides some remarkable tools that can be used with ordinary $\mathbb{R}$−calculus.

This chapter aims to introduce to these ideas. Since theory is particularly complicate, we will do a number of compromises. As usual, proofs will be limited to the simplest and helpful for the understanding. Definitions will be given with some little restriction that helps in presenting the theory quickly.

Chapter requirements: $\mathbb{R}^2$−differential, one variable calculus (differential and integral); vector fields; Gauss-Green formula.

9.2. Elementary functions

In this chapter we deal with functions of complex variable. In this section we introduce a number of non trivial functions that play a fundamental role in the theory. Simple examples of functions $f : D \subset \mathbb{C}$ to $\mathbb{C}$ are powers z^n, combination of powers, that is polynomials $c_0 + c_1 z + \cdots + c_n z^n$, fractions of polynomials, that is *rational functions*. Other simple examples are Re z, Im z, $|z|$, $\bar{z}$.

A general class that encompasses polynomials is the class of *power series* or, improperly, infinite degree polynomials.

Definition 9.2.1

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n (z-w)^n.$$

Here, w is called **centre**, $(c_n) \subset \mathbb{C}$ are the **coefficients**.

Since we have an infinite sum, we have to deal with the problem of convergence of the series. Convergence of $\mathbb{C}-$series works in the same manner of $\mathbb{R}$ series, just the $\mathbb{C}-$modulus takes the place of the $\mathbb{R}-$modulus. In particular, the following *absolute convergence test* holds:

$$\text{if } \sum_{n=0}^{\infty} |a_n| < +\infty, \implies \sum_{n=0}^{\infty} a_n \in \mathbb{C}.$$

Applying the ratio test we can prove the following

Proposition 9.2.2

Assume that $(c_n) \subset \mathbb{C} \backslash \{0\}$ and consider the series

$$\sum_{n=0}^{\infty} c_n (z-w)^n.$$

$$\exists R := \lim_{n \to +\infty} \frac{|c_n|}{|c_{n+1}|} \in [0, +\infty].$$

Then,

- if $R = +\infty$, the series converges (absolutely) for every $z \in \mathbb{C}$;
- if $0 < R < +\infty$, then the series converges absolutely for z such that $|z - w| < R$;
- if $R = 0$, then the series converges only at $z = w$.

Number R is called **radius of convergence** of the series.

Proof. We discuss absolute convergence of the power series, that is convergence for

$$\sum_{n=0}^{\infty} |c_n (z-w)^n| = \sum_{n=0}^{\infty} |c_n| |z-w|^n.$$

This is a constant sign terms series: we apply the ratio test computing

$$q := \lim_{n \to +\infty} \frac{|c_{n+1}| |z-w|^{n+1}}{|c_n| |z-w|^n} = \lim_{n \to +\infty} \frac{|c_{n+1}|}{|c_n|} |z-w| = \frac{|z-w|}{R}.$$

Now, recalling that

- if $q = \frac{|z-w|}{R} < 1$ the series converges (absolutely, hence simply);
- if $q = \frac{|z-w|}{R} > 1$ the series cannot converge;

the conclusion easily follows. □

Example 9.2.3 (Geometric series). *The geometric series* $\sum_{n=0}^{\infty} z^n$ *has radius of convergence* $R = 1$ *and*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ \forall z \in \mathbb{C} \ : \ |z| < 1.$$

Sol. — Here $c_n = 1$ for every n, therefore

$$R = \lim_n \frac{|c_n|}{|c_{n+1}|} = 1.$$

Formula of the sum of the series follows by the same argument when $z \in \mathbb{R}$. □

In the following subsections we introduce some fundamental functions of complex variable.

9.2.1. Exponential. The first remarkable example of function defined through a power series is the *exponential*. The starting point is the identity

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ x \in \mathbb{R}.$$

We use the r.h.s. series to define the exponential for a complex number:

Proposition 9.2.4

Let

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

exp is well defined for every $z \in \mathbb{C}$ and it fulfills the **group identity**

$$\exp(z+w) = \exp(z)\exp(w), \ \forall z, w \in \mathbb{C}. \tag{9.2.1}$$

Moreover,

$$\overline{\exp z} = \exp \bar{z}. \tag{9.2.2}$$

Proof. We have $c_n = \frac{1}{n!}$, thus

$$R = \lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{(n+1)!}{n!} = \lim_n n = +\infty.$$

In particular, the series converges for every $z \in \mathbb{C}$. Let us prove the group identity. We have

$$\exp(z+w) = \sum_n \frac{(z+w)^n}{n!} = \sum_n \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} z^{n-k} w^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!}.$$

On the other side

$$\exp(z)\exp(w) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{j+m=n} \frac{z^j w^m}{j!m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!}.$$

By this, the group identity follows. Next,

$$\overline{\exp z} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \sum_{n=0}^{\infty} \frac{\overline{z^n}}{n!} = \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!} = \exp \bar{z}.$$

□

Since for $x \in \mathbb{R}$ we have $\exp(x) = e^x$, we use the notation

$$e^z := \exp(z), \; z \in \mathbb{C}.$$

Exponential leads naturally to the definition of **hyperbolic functions**,

$$\sinh z := \frac{e^z - e^{-z}}{2}, \quad \cosh z := \frac{e^z + e^{-z}}{2}.$$

Easily, hyperbolic functions are power series,

$$\sinh z = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=2k} \frac{z^n - z^n}{n!} + \sum_{n=2k+1} \frac{z^n + z^n}{n!}\right) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!},$$

and, similarly,

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

9.2.2. Trigonometric functions. From identity (9.2.2) it follows that

$$\left|e^{iy}\right| = 1, \; \forall y \in \mathbb{R}.$$

Indeed, recalling that $|z|^2 = z\bar{z}$, we have

$$\left|e^{iy}\right|^2 = e^{iy}\overline{e^{iy}} = e^{iy}e^{\overline{iy}} = e^{iy}e^{-iy} = e^0 = 1.$$

In particular, e^{iy} is a unitary number that we represent as

$$e^{iy} =: \cos y + i \sin y.$$

Easily, we have the *Euler formulas*

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}. \tag{9.2.3}$$

In particular, we see that

$$\cos y = \cosh(iy), \quad \sin y = -i \sinh(iy).$$

Euler formulas lead to a natural extension of sin and cos to any complex number:

Definition 9.2.5

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}.$$

Notice that, in particular,

$$\cos z = \cosh(iz), \quad \sin z = -i \sinh(iz).$$

By this easily follows the following

Proposition 9.2.6

sin and cos are power series:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \tag{9.2.4}$$

Proof. For example,

$$\cos z = \cosh(iz) = \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} i^{2k} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}. \qquad \square$$

9.2.3. Logarithm. $\mathbb{C}$–logatirhm can be defined but, differently from the previous examples, it is a slightly more complicated function. Since log arises as the inverse of exp, we start with the equation

$$e^z = w,$$

where w is given and z is unknown. If $z = x + iy$,

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y) = e^x u(y),$$

thus

$$e^z = w, \iff e^x u(y) = \rho u(\theta), \iff \begin{cases} e^x = \rho, \\ \\ y = \theta + k2\pi, \ k \in \mathbb{Z}. \end{cases} \iff \begin{cases} x = \log \rho, \\ \\ y = \theta + k2\pi, \ k \in \mathbb{Z}. \end{cases}$$

Thus,

- for $\rho = 0$ (that is $w = 0$), there is no z such that $e^z = 0$ (hence, in particular, $e^z \neq 0$, for every $z \in \mathbb{C}$).
- for $\rho > 0$ (that is $w \neq 0$), *the equation $e^z = w$ has infinitely many solutions*,
$$z_k = \log \rho + i(\theta + k2\pi), \ k \in \mathbb{Z}.$$
These numbers are called **logarithms of** z.

Example 9.2.7. *Solve the following equations:*

i) $e^z = 1$*;*
ii) $e^z = i$*;*
iii) $\sinh z = 0$.
iv) $\cosh z = i$.

Sol. — i),ii) Solutions are

$$z_k = \log|1| + i(\arg 1 + k2\pi) = 0 + ik2\pi = ik2\pi, \ k \in \mathbb{Z},$$

for the first case, while

$$z_k = \log|i| + i(\arg i + k2\pi) = \log 1 + i\left(\frac{\pi}{2} + k2\pi\right) = i\left(\frac{\pi}{2} + k2\pi\right), \ k \in \mathbb{Z}.$$

iii) We write

$$\sinh z = 0, \iff \frac{e^z - e^{-z}}{2} = 0, \iff e^{2z} - 1 = 0, \iff e^{2z} = 1.$$

Solutions are

$$2z_k = \log|1| + ik2\pi, \ k \in \mathbb{Z}, \iff z_k = ik\pi, \ k \in \mathbb{Z}.$$

iv) We have

$$\cosh z = 1, \iff e^z + e^{-z} = 2, \iff e^{2z} + 1 = 2e^z, \iff (e^z - 1)^2 = 0, \iff e^z = 1.$$

Thus

$$z_k = \log|1| + ik2\pi = ik2\pi, \; k \in \mathbb{Z}. \quad \square$$

Example 9.2.8. *Solve*

$$\sin z = i.$$

Sol. — Recalling that $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ we have

$$\sin z = i, \iff e^{iz} - e^{-iz} = 2i^2 = -2, \iff e^{i2z} - 1 = -2e^{iz}, \iff (e^{iz})^2 + 2e^{iz} + 1 = 2,$$

that is

$$(e^{iz} + 1)^2 = 2, \iff e^{iz} + 1 = \pm\sqrt{2}, \iff e^{iz} = -1 \pm \sqrt{2}.$$

Now,

$$e^{iz} = -1 + \sqrt{2}, \iff iz = \log(\sqrt{2} - 1) + ik2\pi, \iff z = 2k\pi - i\log(\sqrt{2} - 1), \; k \in \mathbb{Z}.$$

Similarly

$$e^{iz} = -1 - \sqrt{2}, \iff iz = \log(1 + \sqrt{2}) + i(-\pi + k2\pi), \iff z = -\pi + k2\pi - i\log(1 + \sqrt{2}), \; k \in \mathbb{Z}. \quad \square$$

As for roots, the multiplicity of solutions of $e^z = w$ does not allow to define a funcion

Definition 9.2.9

The **principal logarithm** is the function

$$\log z := \log|z| + i \arg z.$$

Here, $\arg z \in [0, 2\pi[$.

9.3. $\mathbb{C}$−differentiability

In this section we introduce and discuss the concept of $\mathbb{C}$−differentiable function. Differently from the case of functions of several variables, the definition of $\mathbb{C}$−derivative works similarly to the definition of $\mathbb{R}$−derivative.

Definition 9.3.1

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$, D open. We say that f is $\mathbb{C}$−differentiable at point $z \in D$ if

$$\exists \, f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{C}.$$

A function differentiable at every point $z \in D$ is called **holomorphic on** D. We write $f \in H(D)$.

Example 9.3.2. *Powers z^n, $n \in \mathbb{N}$, are holomorphic on $\mathbb{C}$ and the usual relation holds:*

$$(z^n)' = nz^{n-1}, \; \forall n \geqslant 1.$$

Sol. — It is the same calculation we did for real powers. Indeed, according to Newton binomial formula

$$(z+h)^n = z^n + nz^{n-1}h + \sum_{j=2}^{n} \binom{n}{j} z^{n-j} h^j.$$

Therefore

$$\frac{(z+h)^n - z^n}{h} = nz^{n-1} + \sum_{j=2}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) z^{n-j} h^{j-1} \overset{h \longrightarrow 0}{\longrightarrow} nz^{n-1}. \quad \square$$

$\mathbb{C}-$derivative fulfills the same properties of $\mathbb{R}-$derivative, as:

- if $\exists f'(z), g'(z)$ then it exists also $(f \pm g)'(z) = f'(z) \pm g'(z)$.
- if $\exists f'(z), g'(z)$ then it exists also $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$.
- $\exists f'(z), g'(z)$ and $g(z) \neq 0$, then it exists also $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}$.
- if $\exists f'(z)$ and $g'(f(z))$ then there exists also $(g \circ f)'(z) = g'(f(z))f'(z)$.

There is no particular novelty in this, proofs can be extended literally. From these rules some simple functions turns out to be differentiable:

- every polynomial $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ is holomorphic on $\mathbb{C}$.
- every rational function $f(z) := \frac{p(z)}{q(z)}$, where p, q are polynomials, is holomorphic where defined, that is on $D := \{z \in \mathbb{C} \ : \ q(z) \neq 0\}$.

Let us discuss some other interesting examples.

Example 9.3.3. Re z, Im z, $|z|$ *and* $\bar{z}$ *are not differentiable at every point* $z \in \mathbb{C}$.

Sol. — Let us discuss the case of Re z, the others are similar and left as useful exercise. Notice that

$$\frac{\mathrm{Re}\,(z+h) - \mathrm{Re}\,z}{h} = \frac{\mathrm{Re}\,h}{h}.$$

Now, let $z = a + ib$ and $h = x + i0$, $x \in \mathbb{R}$. Clearly,

$$h \longrightarrow 0, \iff x \longrightarrow 0.$$

But then,

$$\frac{\mathrm{Re}\,h}{h} = \frac{x}{x+i0} = 1 \overset{h \longrightarrow 0}{\longrightarrow} 1.$$

Take now $h = 0 + iy$. In this case,

$$\frac{\mathrm{Re}\,h}{h} = \frac{0}{iy} \equiv 0 \longrightarrow 0.$$

Thus, limit

$$\lim_{h \to 0} \frac{\mathrm{Re}\,(z+h) - \mathrm{Re}\,z}{h} \text{ does not exist.} \quad \square$$

Power series are important cases of holomorphic functions.

Theorem 9.3.4

Let $f(z) := \sum_{n=0}^\infty c_n(z-w)^n$ be a power series with radius of convergence $R > 0$. Then

$$\exists f'(z) = \sum_{n=1}^\infty nc_n(z-w)^{n-1},\ \forall z \in \mathbb{C}\ :\ |z-w| < R. \tag{9.3.1}$$

More in general,

$$\exists f^{(k)}(z) = \sum_{n=k}^\infty \frac{n!}{(n-k)!} c_n(z-w)^{n-k},\ \forall z \in \mathbb{C}\ :\ |z-w| < R. \tag{9.3.2}$$

In particular,

$$c_n = \frac{f^{(n)}(w)}{n!}. \tag{9.3.3}$$

The proof is technical and consists in proving that the series can be differentiated term by term, that is,

$$f'(z) = \left(\sum_{n=0}^\infty c_n(z-w)^n\right)' = \sum_{n=1}^\infty nc_n(z-w)^{n-1}.$$

If the sum is finite, this is a consequence of linearity. When the sum is infinite the exchange between sum and derivative is much more delicate. The proof is technical and of no interest here. In particular, we have:

- $(e^z)' = \sum_{n=1}^\infty n\frac{z^{n-1}}{n!} = \sum_{n=1}^\infty \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^\infty \frac{z^n}{n!} = e^z$.
- $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$.
- $(\sin z)' = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!} = \cos z$. Similarly $(\cos z)' = -\sin z$.

Example 9.3.5. *Prove that*

$$\frac{1}{(z-1)^2} = \sum_{n=1}^\infty nz^{n-1},\ \forall |z| < 1.$$

Sol. — Recall that

$$\frac{1}{1-z} = \sum_{n=0}^\infty z^n,\ |z| < 1.$$

Differentiating side by side,

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(\sum_{n=0}^\infty z^n\right)' = \sum_{n=1}^\infty nz^{n-1}. \qquad \square$$

9.4. Cauchy–Riemann conditions

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ and $u = \operatorname{Re} f$, $v := \operatorname{Im} f$ in such a way $f = u + iv$. Real and imaginary part are real valued function of complex variable but, for convenience, we will identify them with functions

$$u = u(x,y),\quad v = v(x,y),$$

or two real variables. In this way

$$f(x+iy) = u(x,y) + iv(x,y).$$

A natural question arises: *what are conditions for u and v such that f be $\mathbb{C}$−differentiable?* We already seen that even extremely simple and nice functions are not $\mathbb{C}$−differentiable. For example, $f(z) = \operatorname{Re} z$. Here, since

$f(x+iy) = x = u + iv$, we see that $u(x,y) = x$ while $v(x,y) \equiv 0$. In particular, u and v are polynomials. Nonetheless, $u+iv$ is not $\mathbb{C}$–differentiable. This leads to suspect that something deep happens. This is the content of the following

Theorem 9.4.1

Let $f = u + iv$. Then, f is $\mathbb{C}$–differentiable if and only if u, v, are $\mathbb{R}^2$–differentiable and they fulfill the following conditions (called **Cauchy-Riemann conditions**):

$$\begin{cases} \partial_x u = \partial_y v \\ \\ \partial_y u = -\partial_x v. \end{cases} \tag{9.4.1}$$

In this case

$$f'(x+iy) = (\partial_x - i\partial_y)u \equiv i(\partial_x - i\partial_y)v.$$

PROOF. Let $z = x+iy$ and $h = \xi + i\eta$. f is differentiable if and only if

$$f(z+h) - f(z) = f'(z)h + o(h).$$

Let $f'(z) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. Then, writing $f = u + iv$, previous identity is equivalent to

$$(u(x+\xi, y+\eta) - u(x,y)) + i\,(v(x+\xi, y+\eta) - v(x,y)) = (\alpha + i\beta)(\xi + i\eta) + o(\xi + i\eta),$$

that is, separating real and imaginary parts and recalling that here $o(\dots)$ is $\mathbb{C}$–valued,

$$\begin{cases} u(x+\xi, y+\eta) - u(x,y) = (\alpha\xi - \beta\eta) + o(\xi,\eta) = (\alpha, -\beta)\cdot(\xi,\eta) + o(\xi,\eta), \\ \\ v(x+\xi, y+\eta) - V(x,y) = (\beta\xi + \alpha\eta) + o(\xi,\eta) = (\beta, \alpha)\cdot(\xi,\eta) + o(\xi,\eta). \end{cases}$$

That is, again: f is $\mathbb{C}$–differentiable iff u and v are differentiable and

$$\nabla u = (\alpha, -\beta), \quad \nabla v = (\beta, \alpha), \quad \Longleftrightarrow \quad \begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

In this case

$$f'(x+iy) = \alpha + i\beta = \partial_x u - i\partial_y u \equiv \partial_y v + i\partial_x v. \qquad \square$$

EXAMPLE 9.4.2. *By using CR conditions, check that $f(z) = \bar{z}$ is never $\mathbb{C}$–differentiable.*

SOL. — If $f(z) = \bar{z} = u + iv$, then $u(x,y) = x$, $v(x,y) = -y$. Clearly u, v are $\mathscr{C}^\infty(\mathbb{R}^2)$. However, since

$$\partial_x u = 1, \quad \partial_y v = -1,$$

the first of the CR conditions is always false. $\square$

EXAMPLE 9.4.3. *The principal logarithm is holomorphic on $D = \mathbb{C}\backslash\mathbb{R}_+$ (where $\mathbb{R}_+ = \{x + i0 \ : \ x \geqslant 0\}$) and*

$$\log'(z) = \frac{1}{z}, \ \forall z \in \mathbb{C}\backslash\mathbb{R}_+.$$

SOL. — Recall that the principal logarithm is defined as

$$\log(z) := \log|z| + i\arg(z) =: u + iv,$$

where,

$$u(x,y) = \log\sqrt{x^2+y^2},\ \ v(x,y) = \arg(x+iy) = \begin{cases} \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right), & y \geqslant 0, \\[2ex] 2\pi - \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right), & y < 0. \end{cases}$$

Notice that

$$\partial_x u = \frac{x}{x^2+y^2},\ \ \partial_y u = \frac{y}{x^2+y^2}$$

while

$$\partial_x v = \begin{cases} -\dfrac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\dfrac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = -\dfrac{y^2}{\sqrt{y^2}(x^2+y^2)} = -\dfrac{y}{x^2+y^2}, & y \geqslant 0, \\[3ex] \dfrac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\dfrac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{y^2}{\sqrt{y^2}(x^2+y^2)} = -\dfrac{y}{x^2+y^2}, & y < 0. \end{cases}$$

By these we deduce that $\partial_x u, \partial_y u \in \mathscr{C}$ thus, according to the total differential thm, u is differentiable. The same holds for v. Moreover,

$$\partial_x v = -\partial_y u$$

and, similarly, $\partial_y v = \partial_x u$. Thus, CR conditions are fulfilles.

Finally,

$$f'(z) = \left(\partial_x - i\partial_y\right) u = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}. \quad \square$$

9.5. Cauchy Theorem

Let $f = u + iv$ be holomorphic on D. According to CR conditions, u, v are differentiable and

$$\partial_x u = \partial_y v,\ \ \partial_y u = -\partial_x v.$$

These conditions looks familiar if we look at them with the language of vector fields. Indeed, consider the vector field $\vec{F} := (v, u)$. Then

$$\partial_x u = \partial_y v \text{ on } D,\ \iff\ \vec{F} := (v,u) \text{ is irrotational on } D.$$

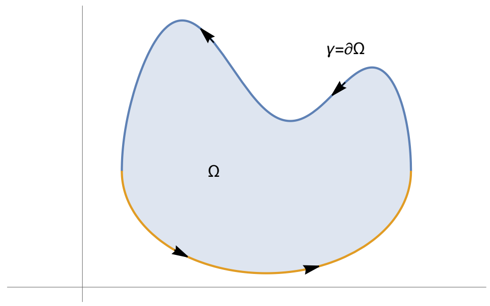
Similarly,

$$\partial_y u = -\partial_x v,\ \text{ on } D,\ \iff\ \vec{G} := (u,-v) \text{ is irrotational on } D.$$

If γ is a counterclockwise oriented circuit such that $\gamma = \partial\Omega$, with $\Omega \subset D$ open, we have

$$\oint_\gamma (v,u) = 0,\ \ \oint_\gamma (u,-v) = 0.$$

We can give a more elegant form to these identities. To this aim, we introduce the



Definition 9.5.1

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function, $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a regular (namely $\mathscr{C}^1$) path, $\gamma \subset D$. We call **path integral** of f along γ the integral

$$\int_\gamma f \equiv \int_\gamma f(\zeta)\, d\zeta := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

If γ is a circuit, namely if $\gamma(a) = \gamma(b)$, the path integral of f along γ is called **circulation** of f along γ and it is denoted by $\oint_\gamma f$.

Few properties of path integrals that will be used frequently. First, if $\gamma = \gamma_1 \cup \gamma_2$, then

$$\int_{\gamma_1 \cup \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

If $\gamma : [a,b] \longrightarrow \mathbb{R}$ is a path, $-\gamma(t) := \gamma(a - t + b)$ is the opposite path of γ. Easily,

$$\int_{-\gamma} f = -\int_\gamma f.$$

An important inequality is the **triangular inequality**:

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\, dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt.$$

We notice also that, if $f = g'$ then

$$\int_\gamma f = \int_\gamma g' = \int_a^b g'(\gamma(t))\gamma'(t)\, dt = g(\gamma(b)) - g(\gamma(a)).$$

In particular, if γ is a circuit, $\gamma(b) = \gamma(a)$, then

$$f = g', \implies \oint_\gamma f = 0.$$

This conclusion actually holds true for every f holomorphic, and this is perhaps the most important result of the theory.

Theorem 9.5.2: Cauchy

Let $f \in H(D)$, and $\gamma \subset D$ a counterclockwise oriented circuit such that $\gamma = \partial\Omega$ with $\Omega \subset D$ open. Then
$$\oint_\gamma f = 0.$$

Proof. Let $f = u + iv$ and $\gamma = \alpha + i\beta$. Then
$$\begin{aligned}\oint_\gamma f &= \int_a^b (u+iv)(\alpha' + i\beta') = \int_a^b (u\alpha' - v\beta') + i\int_a^b (v\alpha' + u\beta')\\ &= \int_a^b (u,-v)\cdot(\alpha',\beta') + i\int_a^b (v,u)\cdot(\alpha',\beta')\\ &= \oint_\gamma (u,-v) + i\oint_\gamma (v,u) = 0. \quad \square\end{aligned}$$

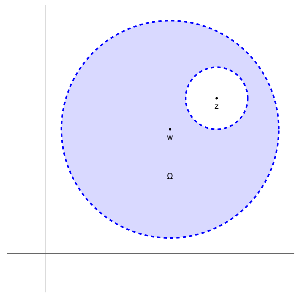
Remark 9.5.3. *What if we compute the circulation of an holomorphic function f along $\gamma \subset D$ but such that $\gamma = \partial\Omega$ with $\Omega \not\subset D$? In general, the circulation might be $\neq 0$. Let us see an example of this.* Consider
$$f(z) := \frac{1}{z}.$$
Certainly $f \in H(\mathbb{C}\backslash\{0\})$. Consider $\gamma = \partial B(0,r[$, with $r > 0$. Certainly $\gamma \subset \mathbb{C}\backslash\{0\}$ but $B(0,r[\not\subset \mathbb{C}\backslash\{0\}$ (because the ball contains 0). Now, considering the standard parametrization for γ,
$$\gamma(t) = r(\cos t + i\sin t) = re^{it},\ t \in [0,2\pi],$$
we have
$$\oint_{\partial B(0,r[} \frac{1}{\zeta}\, d\zeta = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\, dt = i\int_0^{2\pi} 1\, dt = i2\pi \neq 0. \quad \square$$
Previous example is very important. We may extend it to the following formula

Lemma 9.5.4

$$\oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, dz = i2\pi 1_{B(w,r[}(z). \tag{9.5.1}$$

Proof. We have two cases: $z \in B(w,r[$ or $z \in B(w,r]^c$. If $z \in B(w,r[$, let $B(z,\rho[\subset B(w,r[$ and define
$$\Omega := B(w,r[\backslash B(z,\rho] = \{\zeta \in \mathbb{C} \ :\ |\zeta - w| < r,\ |\zeta - z| > \rho\}$$

Since $f(\zeta) := \frac{1}{\zeta - z}$ is $H(\mathbb{C}\backslash\{z\})$ and $\Omega \subset \mathbb{C}\backslash\{z\}$, by the Cauchy theorem we have,
$$\oint_{\partial\Omega} \frac{1}{\zeta - z}\, d\zeta = 0.$$
Now, $\partial\Omega = \partial B(w,r[- \partial B(z,\rho[$, hence
$$0 = \oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta - \oint_{\partial B(z,\rho[} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta - i2\pi,$$



from which (9.5.1) follows for this case.

Second case: $z \in B(w,r]^c$. In this case $B(w,r[\subset \mathbb{C}\backslash\{z\}$. Hence, according to the Cauchy theorem, we have

$$\oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta = 0.$$

From this the conclusion follows. □

Remark 9.5.5. According to the previous Lemma, the quantity

$$\frac{1}{i2\pi}\oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta$$

is 0 or 1 according to $z \in B(w,r[$ or not. The interpretation of this fact is that the integral *counts the number of times the path $\partial B(w,r[$ turns around point z.* □

In a suitable sense, formula (9.5.1) extends to a general representation formula for an holomorphic function:

Theorem 9.5.6: Cauchy formula

Let $f \in H(D)$ and $B(w,r[\subset D$. Then

$$f(z) = \frac{1}{i2\pi}\oint_{\partial B(w,r[} \frac{f(\zeta)}{\zeta - z}\, d\zeta,\ \forall z \in B(w,r[. \tag{9.5.2}$$

Proof. By formula (9.5.1) for $z \in B(w,r[$,

$$\oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta = i2\pi.$$

Then

$$i2\pi f(z) = f(z)\oint_{\partial B(w,r[} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial B(w,r[} \frac{f(z)}{\zeta - z}\, d\zeta,$$

hence the conclusion is equivalent to

$$\oint_{\partial B(w,r[} \frac{f(z)}{\zeta - z}\, d\zeta = \oint_{\partial B(w,r[} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \iff \oint_{\partial B(w,r[} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = 0.$$

Now, let

$$g(\zeta) := \frac{f(\zeta) - f(z)}{\zeta - z}.$$

Certainly, $g \in H(D \backslash \{z\})$ (because of the denominator). Were g holomorphic also at $\zeta = z$, the conclusion would follow by the Cauchy theorem. However, at $\zeta = z$, g is not defined, so we cannot apply the Cauchy theorem. Nonetheless,

$$\lim_{\zeta \to z} g(\zeta) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} = f'(z).$$

Thus, we can extend g by continuity at $\zeta = z$ by posing $g(z) = f'(z)$. We claim that this is sufficient to conclude that

$$\oint_{\partial B(w,r[} g = 0.$$

Indeed, let $B(z, \varepsilon[\subset B(w, r[$ a little ball (ε small). Then, according to Cauchy theorem,

$$\oint_{\partial(B(w,r[\backslash B(z,\varepsilon])} g = 0, \iff \oint_{\partial B(w,r[} g = \oint_{\partial B(z,\varepsilon[} g.$$

Let ε small enough that, by continuity, $|g(\zeta)| \leqslant |f'(z)| + 1$, for every $\zeta \in B(z, \varepsilon[$. Then, according to the triangular inequality,

$$\left| \oint_{\partial B(w,r[} g \right| = \left| \oint_{\partial B(z,\varepsilon[} g \right| \leqslant \int_0^{2\pi} (|f'(z)| + 1) |i\varepsilon e^{it}| \, dt = \varepsilon 2\pi (|f'(z)| + 1).$$

Therefore

$$\left| \oint_{\partial B(w,r[} g \right| \leqslant C\varepsilon,$$

(here $C = 2\pi(|f'(z)| + 1)$). Since ε can be made arbitrarily small, the l.h.s. must equals 0. □

One of the most important consequences of the Cauchy formula is the *analiticity* of an holomorphic function.

Corollary 9.5.7

Let $f \in H(D)$, $D \subset \mathbb{C}$ open. If $B(w, r[\subset D$, then f is sum of a power series convergent on $B(w, r[$. We say that f is **analytic** on D.

Proof. According to the Cauchy formula (9.5.2),

$$f(z) = \frac{1}{i2\pi} \oint_{\partial B(w,r[} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Now, notice that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - w) + (w - z)} = \frac{1}{\zeta - w} \frac{1}{1 - \frac{z-w}{\zeta - w}},$$

and recalling of the geometric sum $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, that holds for $|q| < 1$, we have

$$\frac{1}{1 - \frac{z-w}{\zeta-w}} = \sum_{n=0}^{\infty} \left(\frac{z - w}{\zeta - w} \right)^n,$$

provided

$$\left| \frac{z - w}{\zeta - w} \right| < 1, \iff |z - w| < |\zeta - w| = r,$$

because $\zeta \in \partial B(w, r[$. Hence, returning on the Cauchy formula,

$$f(z) = \frac{1}{i2\pi} \oint_{\partial B(w,r[} \frac{f(\zeta)}{\zeta - w} \sum_{n=0}^{\infty} \left(\frac{z-w}{\zeta - w} \right)^n d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{i2\pi} \oint_{\partial B(w,r[} \frac{f(\zeta)}{(\zeta - w)^{n+1}} d\zeta \right) (z-w)^n \equiv \sum_{n=0}^{\infty} c_n (z-w)^n,$$

as stated[1] □

Remark 9.5.8. *In the proof, we obtained the formula*

$$c_n = \frac{1}{i2\pi} \oint_{\partial B(w,r[} \frac{f(\zeta)}{(\zeta - w)^{n+1}} d\zeta. \tag{9.5.3}$$

Recalling that c_n is also given by (9.3.3), we get the formula,

$$f^{(n)}(w) = \frac{n!}{i2\pi} \oint_{\partial B(w,r[} \frac{f(\zeta)}{(\zeta - w)^{n+1}} d\zeta. \tag{9.5.4}$$

Previous corollary shows that *being $\mathbb{C}-$differentiable in some domain D is completely different form being $\mathbb{R}-$differentiable*. For instance, since a power series is $\mathbb{C}-$differentiable infinitely many times, automatically *a $\mathbb{C}-$differentiable function on D is infinitely many times $\mathbb{C}-$differentiable*! This is completely false with $\mathbb{R}-$differentiability: we may find that f is differentiable, but f' is no more differentiable. Even more: we may see that f might be $\mathscr{C}^\infty$, but yet f is not the sum of any power series. In other words, a function $\mathscr{C}^\infty$ is not necessarily analytic.

Example 9.5.9. *The function*

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is $\mathscr{C}^\infty$ but not analytic.

Sol. — Let us consider $f'(x)$. For $x \neq 0$

$$f'(x) = e^{-1/x^2} \left(\frac{2x}{x^4} \right) = \frac{2}{x^3} e^{-1/x^2}.$$

Easily (comparison exponential vs power at infinity),

$$\lim_{x \to 0} f'(x) = 0,$$

thus $\exists f'(0) = 0$. Similarly one proves that $\exists f^{(n)}(0) = 0$ for every $n \geqslant 2$. If f were analytic, than in some neighbourhood of $x = 0$ we would have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 \equiv 0,$$

which is manifestly false. □

---

[1]There is a delicate passage in the last chain of equalities, that is when we carry out the infinite sum $\sum_{n=0}^{\infty}$ from the integral. It can be proved that this can be done, we skip here the technical details.

9.6. Some consequences of analiticity

In this section we illustrate some of the amazing consequences of the analiticity of an holomorphic function.

Theorem 9.6.1: Liouville

A bounded holomorphic function on $\mathbb{C}$ is necessarily constant.

Proof. Assume $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. We know

$$f(\zeta) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } c_n = \frac{1}{i2\pi} \oint_{\partial B(0,r[} \frac{f(z)}{z^{n+1}}\, dz.$$

By the triangular inequality, for $n \geqslant 1$ we have that,

$$|c_n| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(re^{it})}{(re^{it})^{n+1}} ire^{it} \right| dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{r^{n+1}} r\, dt = \frac{M}{r^n}.$$

Since r is arbitrary, letting $r \longrightarrow +\infty$ we see that $|c_n| \leqslant 0$ for every $n \geqslant 1$. Thus $f(\zeta) = c_0$. □

A remarkable consequence of Liouville theorem is a well known result of Algebra:

Corollary 9.6.2: Fundamental theorem of Algebra

Every non constant polynomial has at least one root in $\mathbb{C}$.

Proof. Assume, by contradiction, that $p(z) \neq 0$, for every $z \in \mathbb{C}$. Let $f(z) := \frac{1}{p(z)}$. Then $f \in H(\mathbb{C})$. Moreover, since p is non constant, easily $|p(z)| \longrightarrow +\infty$ when $|z| \longrightarrow +\infty$. If $m := \inf |p(z)|$ then easily $m > 0$ (otherwise p would vanish). Thus $|f| \leqslant \frac{1}{m}$ is bounded, thus constant according to the Liouville theorem. As a consequence, p itself must be constant. □

Analiticity says that an holomorphic function looks like an "infinity degree" polynomial. As such, zeroes of f behave as for polyomials:

Theorem 9.6.3: zeroes

Let $f \in H(D)$ and w be a zero of f. Then, the following alternative holds:

- either $f \equiv 0$ in some disk $B(w, r[$;
- or, there exists $m \in \mathbb{N}$ and $g \in H(D)$, $g \neq 0$ on some $B(w, r[$ such that
$$f(z) = (z - w)^m g(z).$$
m is called **multiplicity of f at w** and it is denoted by $m(f, w)$.

Proof. Let $B(w, r[\subset D$ on which

$$f(z) = \sum_{n=0}^{\infty} c_n (z - w)^n.$$

Then,

- either $c_n = 0$ for all $n \in \mathbb{N}$, hence $f \equiv 0$ on $B(w, r[$,

- or $c_0 = c_1 = \ldots = c_{m-1} = 0$ and $c_m \neq 0$ for some $m \geqslant 1$. In particular, if we set
$$g(z) := \frac{f(z)}{(z-w)^n},$$
certainly $g \in H(D\backslash\{w\})$ and since
$$f(z) = \sum_{n=m}^{\infty} c_n (z-w)^n, \implies g(z) = \sum_{n=m}^{\infty} c_n (z-w)^{n-m} = \sum_{k=0}^{\infty} c_{k+m}(z-w)^k.$$
By this it follows that g can be extended by continuity also at $z = w$ taking $g(w) = c_m \neq 0$. Finally, being $g(w) \neq 0$, by continuity and possibly reducing r, we can assume $g \neq 0$ on $B(w, r[$.

Notice that, reading carefully the proof of previous proposition, we have the formula
$$m(f, w) = \min\{k \ : \ f^{(k)}(w) \neq 0\}$$

Example 9.6.4. *Determine all zeroes of $f(z) = \cosh z$ with their multiplicity.*
$$\cosh z = 0, \iff \frac{e^z + e^{-z}}{2} = 0, \iff e^{2z} + 1 = 0, \iff e^{2z} = -1,$$
that is, recalling of complex logarithm,
$$2z = \log|-1| + i\,(\pi + k2\pi)\,,\ k \in \mathbb{Z}, \iff z_k = i\left(\frac{\pi}{2} + k\pi\right),\ k \in \mathbb{Z}.$$
Now, $f'(z) = \sinh z$ and
$$f'(z_k) = \sinh\left(i\left(\frac{\pi}{2} + k\pi\right)\right) = i\sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k i.$$
Since $f(z_k) = 0$ and $f'(z_k) = (-1)^k i \neq 0$ we conclude that $m(z_k) = 1$ for every k. □

Remark 9.6.5. *Once more, we underline the difference between $\mathbb{R}-$differentiable and $\mathbb{C}-$differentiable functions. In the example 9.5.9, we have a function f having a zero at $x = 0$, but all its derivatives $f^{(k)}(0) = 0$. Nonetheless, apart for $x = 0$, f is always $\neq 0$.* □

9.7. Isolated singularities

According to the Cauchy theorem, in a neighbourhood of a point w where a function f is $\mathbb{C}-$differentiable, the function is a power series, that is,
$$f(z) = \sum_{n=0}^{\infty} c_n (z-w)^n.$$
Suppose now that f be holomorphic around a point w without being $\mathbb{C}-$differentiable at w. For example,
$$\frac{1}{z-w} \in H(\mathbb{C}\backslash\{w\}).$$
More in general, a function of type
$$\sum_{n=1}^{N} \frac{d_n}{(z-w)^n} \in H(\mathbb{C}\backslash\{w\}).$$
As power series centred at w are prototypes of holomorphic functions at w, for a function not holomorphic at w we are naturally led to consider a function of type
$$\sum_{n=1}^{\infty} \frac{d_n}{(z-w)^n} \equiv \sum_{n=1}^{\infty} c_{-n}(z-w)^{-n}.$$

Combining these with power series we have a new class of functions, enlarging that one made by power series.

Definition 9.7.1

A **bilateral power series** is a series of type

$$\sum_{n=-\infty}^{\infty} c_n(z-w)^n := \underbrace{\sum_{n=0}^{\infty} c_n(z-w)^n}_{regular\ part} + \underbrace{\sum_{n=1}^{\infty} c_{-n}(z-w)^{-n}}_{singular\ part}.$$

Convergence of a bilateral series is easy. Since the *regular part* is an ordinary power series, it converges on a disk $B(w,R[$ for some $R>0$ (we do not consider here the case when $R=0$ because it is of no interest). On the other hand, the singular part is itself a power series of $(z-w)^{-1}$, thus it will converge when $|z-w|^{-1}<r$ for some $r>0$, that is for $|z-w|>\frac{1}{r}$. The conclusion is that

$$\text{a bilateral series converges on a set of type } \{z\in\mathbb{C}\ :\ \rho<|z-w|<R\},$$

which is an **annulus**. If $\rho>R$ clearly this set is empty.

Bilateral series plays the role played by power series for holomorphic functions when the function has a *isolated singularity* at $z=w$. It is convenient to introduce a notation:

$$B_*(w,r[:=B(w,r[\backslash\{w\}$$

We call B_* a **punctured neighbourhood** of w.

Definition 9.7.2

Let $w\in\mathbb{C}$. We say that w is an **isolated singularity** for a function f if

$$f\in H(B_*(w,r[),\ \text{for some } r>0.$$

As annouched,

Theorem 9.7.3: Laurent

If w is an isolated singularity for f then, in a punctured neighbourhood of w, f is a bilateral series. This series is called **Laurent series**.

Proof. The proof is similar to the proof of the analyticity for an holomorphic function. Let $f\in H(B_*(w,r[)$. Let $z\in B_*(w,r[$, R_1,R_2 such that

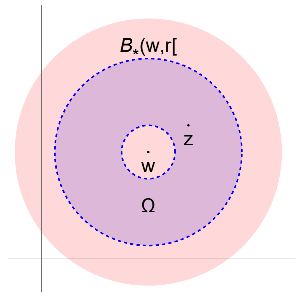
$$z\in\Omega:=\{\zeta\in\mathbb{C}\ :\ R_1<|\zeta-w|<R_2\}\subset B_*(w,r[.$$

Let

$$g(\zeta):=\frac{f(\zeta)-f(z)}{\zeta-z},$$

extended by continuity at $\zeta=z$ with value $f'(z)$. Being f analytic around z,

$$f(\zeta)=f(z)+\sum_{n=1}^{\infty}c_n(\zeta-z)^n,\ \Longrightarrow\ g(\zeta)=\sum_{n=0}^{\infty}c_{n+1}(\zeta-z)^n,$$



thus g is analytic at $\zeta = z$. According to Cauchy theorem

$$\oint_{\partial\Omega} g = 0, \iff \oint_{\partial B(w,R_1[} g = \oint_{\partial B(w,R_2[} g.$$

We start with the l.h.s. of previous identity. We have

$$\oint_{\partial B(w,R_1[} g = \oint_{\partial B(w,R_1[} \frac{f(\zeta)-f(z)}{\zeta - z}\, d\zeta = \oint_{\partial B(w,R_1[} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\oint_{\partial B(w,R_1[} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial B(w,R_1[} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

because, according formula (9.5.1), $\oint_{\partial B(w,R_1[} \frac{1}{\zeta - z}\, d\zeta = 0$. Now,

$$\frac{1}{z-\zeta} = \frac{1}{(z-w)-(\zeta - w)} = \frac{1}{z-w}\frac{1}{1-\frac{\zeta-w}{z-w}} = \frac{1}{z-w}\sum_{n=0}^{\infty}\left(\frac{\zeta-w}{z-w}\right)^n$$

which is convergent because $\left|\frac{\zeta-w}{z-w}\right| = \frac{R_1}{|z-w|} < 1$. Thus

$$\oint_{\partial B(w,R_1[} g = \sum_{n=0}^{\infty}\left(-\oint_{\partial B(w,R_1[} f(\zeta)(\zeta-w)^n\, d\zeta\right)(z-w)^{-n-1} = \sum_{n=1}^{\infty} c_{-n}(z-w)^{-n}.$$

Similarly,

$$\oint_{\partial B(w,R_2[} g = \oint_{\partial B(w,R_2[} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\oint_{\partial B(w,R_2[} \frac{1}{\zeta - z}\, d\zeta = \oint_{\partial B(w,R_1[} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) i2\pi.$$

With the same argument of analiticity theorem,

$$\oint_{\partial B(w,R_1[} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} c_n (z-w)^n,$$

thus

$$i2\pi f(z) = \sum_{n=0}^{\infty} c_n (z-w)^n - \sum_{n=1}^{\infty} c_{-n}(z-w)^{-n},$$

from which the conclusion follows. □

The calculation of a Laurent series for a given function might not be an entirely easy task. Indeed, differently from points where a function is holomorphic for which the power series coefficients can be determined through the formulas (9.3.3), for a bilateral series these formula do not make sense.

The shape of the singular part of the Laurent expansion leads to classify the isolated singularities.

Definition 9.7.4

Let w be an isolated singularity for f,

$$f(z) = \sum_{n=1}^{\infty} c_{-n}(z-w)^{-n} + \sum_{n=0}^{\infty} c_n (z-w)^n.$$

We say that w is

- a **removable singularity** if the singular part of Laurent series vanishes, that is $c_{-n} = 0$ for every $n \geqslant 1$;
- a **pole of order** k if $c_{-n} = 0$ for every $n \geqslant k+1$ and $c_{-k} \neq 0$;
- an **essential singularity** if $c_{-n} \neq 0$ for infinitely many n.

The coefficient c_{-1} is called **residue of** f **at** w and it is denoted by $\mathrm{Res}(f, w)$.

Let us show some examples of these singularities.

Example 9.7.5. *Let $f(z) = \frac{e^z - 1}{z}$. Then $z = 0$ is a removable singularity.*

Sol. — Clearly $f \in H(\mathbb{C}\backslash\{0\})$, hence 0 is an isolated singularity. To determine the Laurent expansion of f we remind that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \longrightarrow \quad f(z) = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) = \frac{1}{z}\sum_{n=1}\frac{z^n}{n!} = \underbrace{0}_{sing.\ part} + \underbrace{\sum_{n=0}^{\infty} \frac{z^n}{(n-1)!}}_{reg.part}, \ \forall z \neq 0.$$

Since this series contains only the regular part it means that its singular part is identically 0. This means that 0 is a **removable singularity**. □

Example 9.7.6. *Let $f(z) = \frac{e^z - 1}{z^{k+1}}$. Then $z = 0$ is a pole of order k.*

Sol. — We proceed similarly to the previous example:

$$f(z) = \frac{1}{z^{k+1}}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \underbrace{\frac{1}{z^k} + \frac{1}{2!z^{k-1}} + \cdots + \frac{1}{(k-1)!z}}_{sing.\ part} + \underbrace{\sum_{n=0}^{\infty}\frac{z^n}{(n+k)!}}_{reg.\ part}.$$

from which we see that $z = 0$ is a **pole of order** k. □

Example 9.7.7. *Let $f(z) = e^{1/z}$. Then $z = 0$ is an essential singularity.*

Sol. — Again, by the exponential series,

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{(1/z)^n}{n!} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{n!z^n}}_{sing.\ part} + \underbrace{1}_{reg.\ part}.$$

By this we see that 0 is an **essential singularity** for f. □

An important case is when

$$f(z) = \frac{N(z)}{D(z)}, \quad \text{where } N, D \in H(\mathbb{C}).$$

Clearly

$$f \in H(\mathbb{C}\backslash\{D = 0\}).$$

Thus, the singularities of f are the zeroes of D. According to zeroes theorem, unless $D \equiv 0$, zeroes of D are isolated, that is singularities of f are isolated. Let w be a zero of D. From zeroes theorem, w has a certain multiplicity $m(D, w)$ as zero for D. Let $m(N, w)$ the multiplicity of zero for N (if $N(w) \neq 0$ then $m(N, w) = 0$). By the zeroes theorem

$$f(z) = \frac{(z - w)^{m(N,w)} g(z)}{(z - w)^{m(D,w)} h(z)} = \frac{1}{(z - w)^{m(D,w)-m(N,w)}} \frac{g(z)}{h(z)} =: \frac{1}{(z - w)^{m(D,w)-m(N,w)}} \varphi(z),$$

where φ is holomorphic and $\varphi(w) \neq 0$. By this it follows that w *is*

- a **pole of order** $m(D, w) - m(N, w)$ if $m(D, w) - m(N, w) > 0$;
- a **removable singularity** if $m(D, w) - m(N..w) \leqslant 0$.

EXAMPLE 9.7.8. *Determine and classify the singularities of the function*

$$f(z) := \frac{1}{e^{iz} - 1}.$$

SOL. — $f = N/D$ where $N \equiv 1$, $D = e^{iz} - 1$, $N, D \in H(\mathbb{C})$. We have

$$D(z) = 0, \iff e^{iz} = 1, \iff iz = \log 1 + ik2\pi,\ k \in \mathbb{Z}, \iff z_k = k2\pi,\ k \in \mathbb{Z}.$$

Since $D'(z) = ie^{iz}$, $D'(k2\pi) = ie^{k2pi} = i \neq 0$ for every $k \in \mathbb{Z}$, thus $m(D, z_k) = 1$. Since $m(N, z_k) = 0$, we have $m(D, z_k) - m(N, z_k) = 1$, that is z_k is a pole of order 1. □

9.8. Residues Theorem

According to the Cauchy theorem, if $f \in H(D)$ and $\gamma = \partial\Omega$ with $\Omega \subset D$ open,

$$\oint_\gamma f = 0.$$

In this section we extend this result to the case when Ω contains a certain number of isolated singularities.

Theorem 9.8.1

Let $f \in H(\mathbb{C}\backslash\{w_1, \ldots, w_N\})$, $\gamma \subset \mathbb{C}\backslash\{w_1, \ldots, w_N\}$ a counterclockwise oriented circuit such that $\gamma = \partial D$, with D open. Then

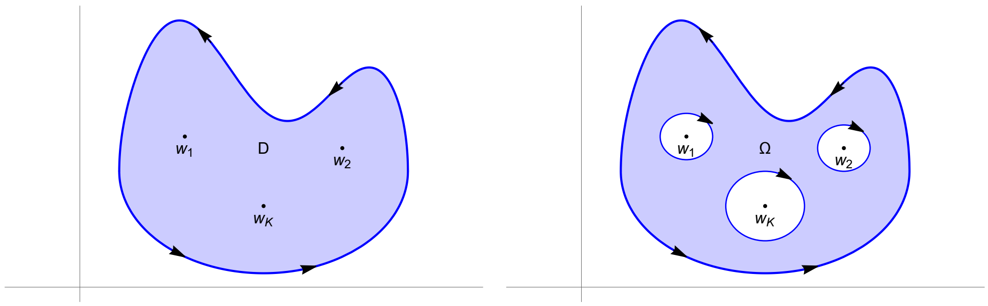
$$\oint_\gamma f = i2\pi \sum_{w_k \in D} \text{Res}(f, w_k). \tag{9.8.1}$$

PROOF. We assume that $w_1, \ldots, w_K \in D$ and $w_{K+1}, \ldots, w_N \notin D$. Let ε small enough such that $B(w_j, \varepsilon[\subset D$, $j = 1, \ldots, K$, and

$$\gamma \cap \partial B(w_j, \varepsilon[= \emptyset.$$

Let also

$$\Omega := D \backslash \bigcup_{j=1}^{K} B(w_j, \varepsilon[.$$



Since $f \in H(\Omega)$, according to the Cauchy theorem

$$\oint_{\partial\Omega} f = 0,$$

that is

$$\oint_{\gamma} f - \sum_{j=1}^{K} \oint_{\partial B(w_j,\varepsilon[} f = 0.$$

According to Laurent theorem, on $B_*(w_j,\varepsilon[$, $f(z) = R_j(z) + S_j(z)$ where $R_j \in H(B(w_j,\varepsilon[)$ is the regular part,

$$S_j(z) = \sum_{n=1}^{\infty} c_{-n,j}(z - w_j)^{-n}$$

is the singular part. Again, by Cauchy theorem $\oint_{\partial B(w_j,\varepsilon[} R_j = 0$, thus

$$\oint_{\partial B(w_j,\varepsilon[} f = \oint_{\partial B(w_j,\varepsilon[} S_j = \oint_{\partial B(w_j,\varepsilon[} \sum_{n=1}^{\infty} c_{-n,j}(z - w_j)^{-n}\, dz = \sum_{n=1}^{\infty} c_{-n,j} \oint_{\partial B(w_j,\varepsilon[} (z - w_j)^{-n}\, dz.$$

For $n \geqslant 2$,

$$(z - w_j)^{-n} = \left(\frac{(z - w_j)^{-n+1}}{-n+1}\right)', \implies \oint_{\partial B(w_j,\varepsilon[} (z - w_j)^{-n}\, dz = 0,$$

while, for $n = 1$, according to formula (9.5.1)

$$\oint_{\partial B(w_j,\varepsilon[} (z - w_j)^{-1}\, dz = \oint_{\partial B(w_j,\varepsilon[} \frac{1}{z - w_j}\, dz = i2\pi.$$

Therefore

$$\oint_{\gamma} f = \sum_{j=1}^{K} i2\pi c_{-1,j},$$

which is the conclusion. □

The *residues theorem* points out the importance of the coefficient c_{-1} of the first negative power in the Laurent expansion around an isolated pole. It is important to have efficient methods to compute the residue.

Proposition 9.8.2

Let w be a pole of order k for f. Then

$$\operatorname{Res}(f,w) = \frac{1}{(k-1)!}\lim_{z\to w}\frac{d^{k-1}}{dz^{k-1}}(z-w)^k f(z). \tag{9.8.2}$$

Proof. If w is a pole of order k, we have

$$f(z) = \sum_{n=0}^{\infty} c_n(z-w)^n + \sum_{n=1}^{k}\frac{c_{-n}}{(z-w)^n},$$

with $c_{-k} \neq 0$. Then

$$(z-w)^k f(z) = \sum_{n=0}^{\infty} c_n(z-w)^{n+k} + \sum_{n=1}^{k} c_{-n}(z-w)^{k-n} =: g(z). \tag{9.8.3}$$

On the right side we have an analytic function and c_{-1} is the coefficient of $(z-w)^{k-1}$. Therefore,

$$c_{-1} = \frac{g^{(k-1)}(w)}{(k-1)!}.$$

Since f is not defined at w, we cannot replace g with $(\sharp - w)^k f(\sharp)$, derive and evaluate at w. However, by (9.8.3), for $z \neq w$ we can write

$$\frac{d^{k-1}}{dz^{k-1}}(z-w)^k f(z) = g^{(k-1)}(z) \longrightarrow g^{(k-1)}(w) = (k-1)!c_{-1}, \text{ when } z \longrightarrow w,$$

and this is exactly the (9.8.2). □

A particular case is $f = N/D$. If $m(N,w) = 0$ and $m(D,w) = 1$. In this case w is an isolated pole of order 1. Therefore, by (9.8.2) we have

$$\operatorname{Res}(f,w) = \lim_{z\to w}(z-w)f(z) = \lim_{z\to w}(z-w)\frac{N(z)}{D(z)} = \lim_{z\to w}\frac{N(z)}{\frac{D(z)-D(w)}{z-w}} = \frac{N(w)}{D'(w)}. \tag{9.8.4}$$

9.9. Applications to generalized integrals

The Cauchy Theorem and the Residues Theorem apply fruitfully to the computation of integrals on the real line. In this section, we discuss this application in several notable cases.

9.9.1. Rational functions. An algorithm is well known to compute

$$\int_{-\infty}^{+\infty}\frac{p(x)}{q(x)}\,dx,$$

where p, q are polynomials. This integral converges iff $\deg(q) \geqslant \deg(p) + 2$ and the standard calculation is based on the calculation of a *primitive* for p/q. Here, we show an alternative and faster method based on complex integration.

The idea is the following. Let f be a rational function, that is,

$$f(z) := \frac{p(z)}{q(z)}, \quad p, q \text{ polynomials.}$$

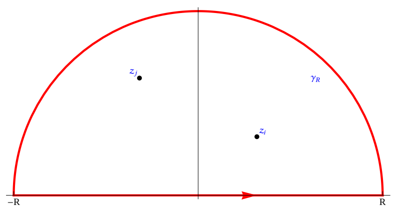
This function f is holomorphic on $\mathbb{C}\backslash\{q=0\}$, and since q is a polynomial, this means $f \in H(\mathbb{C}\backslash\{w_1,\dots,w_N\})$, where w_j are zeroes for q. We assume that none of these zeroes belongs to $\mathbb{R}$. Then

$$\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx = \lim_{r\to+\infty}\int_{-r}^{r}\frac{p(x)}{q(x)}\,dx = \lim_{r\to+\infty}\int_{[-r,r]} f.$$

Define

$$\gamma_r := [-r,r]+\sigma_r,$$

where $\sigma_r : [0,\pi] \longrightarrow \mathbb{C}$, $\sigma_r(t) = re^{it}$. For r large enough, $\gamma_r = \partial\Omega$ with $\Omega \supset S_+ := \{w_j \ : \ \mathrm{Im}\, w_j > 0\}$. Now, by



the residues theorem,

$$\oint_{\gamma_r} f = i2\pi \sum_{w\in S_+} \mathrm{Res}(f,w).$$

Notice that the r.h.s. does not depend on r , then

$$\int_{-\infty}^{+\infty}\frac{p(x)}{q(x)}\,dx = \lim_{r\to+\infty}\int_{-r}^{r}\frac{p(x)}{q(x)}\,dx = -\lim_{r\to+\infty}\int_{\sigma_r} f + i2\pi\sum_{w\in S_+}\mathrm{Res}(f,w).$$

We claim that the contribution on σ_r vanishes as $r \longrightarrow +\infty$. Indeed, by the triangular inequality of path integrals,

$$\left|\int_{\sigma_r} f\right| \leqslant \int_0^{\pi} |f(re^{i\theta})||ire^{i\theta}|\,d\theta = r\int_0^{\pi}|f(re^{i\theta})|\,d\theta.$$

Now, since $\deg(q) \geqslant \deg(p)+2$, we have that, for $z \longrightarrow \infty_{\mathbb{C}}$,

$$|f(z)| = \frac{|p(z)|}{|q(z)|} \sim \frac{a|z|^{\deg p}}{b|z|^{\deg q}} = \frac{c}{|z|^{\deg q-\deg p}} \leqslant \frac{c}{|z|^2},$$

thus

$$r\int_0^{\pi}|f(re^{i\theta})|\,d\theta \leqslant r\int_0^{\pi}\frac{c}{|re^{i\theta}|^2}\,d\theta = c\frac{r}{r^2}2\pi = \frac{2\pi c}{r} \longrightarrow 0.$$

Conclusion:

(9.9.1)
$$\int_{-\infty}^{+\infty}\frac{p(x)}{q(x)}\,dx = i2\pi\sum_{w\in S_+}\mathrm{Res}\left(\frac{p}{q},w\right).$$

Example 9.9.1. *Compute*

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^4}\,dx.$$

Sol. — Let $f(x) := \frac{1}{1+x^4}$. Clearly $f \in \mathscr{C}(\mathbb{R})$ and because $f \sim_{\pm\infty} \frac{1}{x^4}$ it is integrable at $\pm\infty$. Thus, the proposed integral converges. The singularities of f are the zeroes of $q(w) = 1+w^4$. Now, $q(w)=0$ iff $w^4=-1$, that is $w_k = e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}$, $k = 0,1,2,3$. Now, by (9.9.1)

$$\int_{-\infty}^{+\infty}\frac{1}{1+x^4}\,dx = i2\pi\left(\mathrm{Res}(f,e^{i\frac{\pi}{4}}) + \mathrm{Res}(f,e^{i\frac{3\pi}{4}})\right).$$

Let us classify the singularities. Because $p \equiv 1$ clearly $m(p, w_k) = 0$. Moreover $q'(z) = 4z^3$, therefore $q'(w_k) \neq 0$ for any k. Thus $m(q, w_k) = 1$, and by this we deduce that each w_k is a first order pole for f. To compute the residue at any w_k we can apply the reduced formula (9.8.4):

$$\operatorname{Res}(f, w_k) = \frac{p(w_k)}{q'(w_k)} = \frac{1}{4w_k^3} = \frac{1}{4}e^{-i\frac{3}{4}\pi}, \frac{1}{4}e^{-i\frac{\pi}{4}},\ k = 0, 1,$$

hence

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\, dx = i2\pi \left(\frac{1}{4}e^{-i\frac{3}{4}\pi} + \frac{1}{4}e^{-i\frac{\pi}{4}}\right) = i\frac{\pi}{2}\left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) = -i\frac{\pi}{2}\frac{2}{\sqrt{2}}i = \frac{\pi}{\sqrt{2}}. \quad \square$$

Example 9.9.2. *Compute*

$$\int_0^{+\infty} \frac{x^2+3}{(x^2+1)(x^2+4)}\, dx.$$

Sol. — Let $f(x) := \frac{x^2+3}{(x^2+1)(x^2+4)}$. Clearly $f \in \mathscr{C}(\mathbb{R})$, and $f(x) \sim_{\pm\infty} \frac{x^2}{x^4} = \frac{1}{x^2}$, f is absolutely integrable on $\mathbb{R}$. Moreover, since f is even,

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\, dx.$$

To compute this we apply the residue theorem. The singular points of $f(z) = \frac{z^2+3}{(z^2+1)(z^2+4)}$ are $z = \pm i$ and $z = \pm 2i$. Writing

$$f(z) = \frac{(z-3i)(z+3i)}{(z-i)(z+i)(z-2i)(z+2i)},$$

we see immediately that all the singular points are first order poles. Therefore, by (9.9.1)

$$\int_{-\infty}^{+\infty} f(x)\, dx = i2\pi\left(\operatorname{Res}(f, i) + \operatorname{Res}(f, 2i)\right).$$

To compute the residues, it is easier by using the general formula (9.8.2).

$$\operatorname{Res}(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-3i)(z+3i)}{(z+i)(z-2i)(z+2i)} = \frac{(-2i)(4i)}{(2i)(-i)(3i)} = -\frac{4}{3}i,$$

and

$$\operatorname{Res}(f, 2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{(z-3i)(z+3i)}{(z-i)(z+i)(z+2i)} = \frac{(-i)(5i)}{(i)(3i)(4i)} = \frac{5}{12}i.$$

Therefore

$$\int_{-\infty}^{+\infty} f(x)\, dx = i2\pi\left(-\frac{4}{3}i + \frac{5}{12}i\right) = \frac{11}{6}\pi. \quad \square$$

The next example is not an application of formula (9.9.1), nonetheless it involves the same order of ideas.

Example 9.9.3. *Compute*

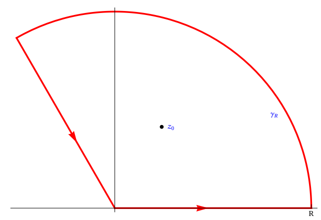
$$\int_0^{+\infty} \frac{1}{1+x^3}\, dx$$

Sol. — Clearly the integral exists finite. Let $f(z) = \frac{1}{1+z^3}$, $f \in H(\mathbb{C}\backslash\{e^{i\frac{\pi}{3}+ik\frac{2\pi}{3}} \ :\ k = 0, 1, 2\})$. It is evident that all the singularities of f are first order poles. Therefore, if

$$\gamma_r := [0, r] + \sigma_r + \left[re^{i\frac{2\pi}{3}}, 0\right], \text{ where } \sigma_r(t) = re^{it},\ t \in \left[0, \frac{2\pi}{3}\right].$$

by the residue theorem we have,

$$\oint_{\gamma_r} f = i2\pi \mathrm{Res}\left(f, e^{i\frac{\pi}{3}}\right) = i2\pi \frac{1}{3e^{i\frac{2\pi}{3}}}.$$



We have $\oint_{\gamma_r} = \int_{[0,r]} + \int_{\sigma_r} + \int_{[re^{i\frac{2\pi}{3}},0]}$. Notice that

$$\int_{\left[re^{i\frac{2\pi}{3}},0\right]} f = -\int_0^r f\left(te^{i\frac{2\pi}{3}}\right) e^{i\frac{2\pi}{3}}\, dt = -e^{i\frac{2\pi}{3}} \int_0^r \frac{1}{1+\left(te^{i\frac{2\pi}{3}}\right)^3}\, dt = -e^{i\frac{2\pi}{3}} \int_0^r \frac{1}{1+t^3}\, dt.$$

Moreover, by the triangular inequality

$$\left|\int_{\sigma_r} f\right| \leqslant \int_0^{\pi/3} |f(re^{i\theta})||ire^{i\theta}|\underset{.}{\theta} = r\int_0^{\pi/3} \frac{1}{|r^3e^{i3\theta}+1|\, d\theta.}$$

Since $r \longrightarrow \grave{e}\infty$, $|r^3e^{i3\theta}+1| \geqslant |r^3e^{i3\theta}| - 1 = r^3 - 1$, thus

$$\left|\int_{\sigma_r} f\right| \leqslant \frac{r}{r^3-1}\frac{\pi}{3} \longrightarrow 0.$$

Therefore

$$\left(1 - e^{i\frac{2\pi}{3}}\right)\int_0^{+\infty} \frac{1}{1+t^3}\, dt = i2\pi \frac{1}{3e^{i\frac{2\pi}{3}}},$$

that is

$$\int_0^{+\infty} \frac{1}{1+t^3}\, dt = \frac{i2\pi}{3e^{i\frac{2\pi}{3}}\left(1-e^{i\frac{2\pi}{3}}\right)} \frac{e^{-i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}} = -\frac{\pi}{3e^{i\pi}\frac{e^{i\frac{\pi}{3}}-e^{-i\frac{\pi}{3}}}{2i}} = \frac{\frac{\pi}{3}}{\sin\frac{\pi}{3}}. \qquad \square$$

9.9.2. Fourier integrals. A *Fourier integral* is an integral of type

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx,\ \xi \in \mathbb{R}.$$

The integral is convergent if $f \in L^1(\mathbb{R})$ because $|e^{i\xi x}f(x)| = |f(x)|$ for any $x \in \mathbb{R}$. We assume that f is actually defined on $\mathbb{C}\backslash S$, where S is a finite set of poles. We can proceed as in the previous case writing

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = \lim_{r\to+\infty} \int_{-r}^{r} e^{i\xi x} f(x)\, dx = \lim_{r\to+\infty} \int_{[-r,r]} e^{i\xi z} f(z)\, dz.$$

Notice that if $f \in H(\mathbb{C}\backslash S)$ then $e^{i\xi\sharp}f(\sharp) \in H(\mathbb{C}\backslash S)$ and of course if $w_j \in S$ is a pole for f, it is also a pole for $e^{i\xi\sharp}f(\sharp)$ of the same order (exercise). Therefore, considering $\gamma_r := [-r,r] + \sigma_r$ where $\sigma_r(t) = re^{it}$, $t \in [0,\pi]$ we have, as r is big enough, by the residue theorem

$$\oint_{\gamma_r} e^{i\xi z} f(z)\, dz = i2\pi \sum_{w\in S_+} \mathrm{Res}\left(e^{i\xi\sharp}f(\sharp), w\right),$$

where $S_+ = S \cap \{\mathrm{Im}\, z > 0\}$. Because $\oint_{\gamma_r} = \int_{[-r,r]} + \int_{\sigma_r}$, the main problem is computing the limit

$$\lim_{r\to+\infty} \int_{\sigma_r} e^{i\xi z} f(z)\, dz.$$

Notice that

$$|e^{i\xi z} f(z)| = |f(z)| e^{-\xi \mathrm{Im}\, z}.$$

Hence, if $\xi \geqslant 0$, being $\sigma_r \subset \{\mathrm{Im}\, z > 0\}$, we have

$$\left| \int_{\sigma_r} e^{i\xi z} f(z)\, dz \right| \leqslant \int_0^{2\pi} |f(re^{i\theta})||ire^{i\theta}|\, d\theta = \int_0^{2\pi} |re^{i\theta} f(re^{i\theta})|\, d\theta.$$

Assume now that

$$\lim_{z\to\infty} z f(z) = 0.$$

Then, for $|z| \geqslant r$, $|zf(z)| \leqslant \varepsilon$, hence

$$\left| \int_{\sigma_r} e^{i\xi z} f(z)\, dz \right| \leqslant 2\pi\varepsilon,$$

that is

$$\lim_{r\to+\infty} \int_{\sigma_r} e^{i\xi z} f(z)\, dz = 0.$$

Thus, we can conclude that

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = i2\pi \sum_{w\in S_+} \left(e^{i\xi\sharp} f(\sharp), w \right),\ \forall \xi \geqslant 0. \tag{9.9.2}$$

If $\xi \leqslant 0$ this argument fails because $e^{-\xi \mathrm{Im}\, z}$ is unbounded. However, this can be easily fixed by changing γ_r with $\widetilde{\gamma}_r := [-r,r] + (-\widetilde{\sigma}_r)$ where $\widetilde{\sigma}_r(t) = re^{it}$, $t \in [\pi, 2\pi]$. For r is big enough, the residue theorem gives

$$\oint_{\widetilde{\gamma}_r} e^{i\xi z} f(z)\, dz = -i2\pi \sum_{w\in S_-} \mathrm{Res}\left(e^{i\xi\sharp} f(\sharp), w \right),$$

where $S_- = S \cap \{\mathrm{Im}\, z < 0\}$ and there's the $-$ because the path is clockwise oriented.

Example 9.9.4. *Compute*

$$\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{x^2 - 2x + 2}\, dx,\ \xi \in \mathbb{R}.$$

Sol. — It is easy to check that $f(x) := \frac{1}{x^2-2x+2}$ is integrable. Indeed: $f \in \mathscr{C}(\mathbb{R})$ and $|f(x)| \sim_{\pm\infty} \frac{1}{x^2}$. Of course f is defined on $\mathbb{C}\backslash\{z^2 - 2z + 2 = 0\}$, that is on $\mathbb{C}\backslash\{w_1, w_2\}$ where

$$w_{1,2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2} = 1 \pm i.$$

Clearly these are poles of first order. Moreover

$$\lim_{z\to\infty} z f(z) = 0,$$

therefore

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = \begin{cases} i2\pi \mathrm{Res}\left(e^{i\xi\sharp} f(\sharp), 1+i \right), & \xi \geqslant 0, \\ \\ -i2\pi \mathrm{Res}\left(e^{i\xi\sharp} f(\sharp), 1-i \right), & \xi < 0. \end{cases}$$

Easily we get

$$\operatorname{Res}\left(e^{i\xi\sharp}f(\sharp),1+i\right)=\left.\frac{e^{i\xi z}}{2z-2}\right|_{z=1+i}=\frac{e^{i\xi-\xi}}{2i},\ \ \operatorname{Res}\left(e^{i\xi\sharp}f(\sharp),1-i\right)=-\frac{e^{i\xi+\xi}}{2i},$$

that is

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{x^2-2x+2}\,dx=\pi e^{-|\xi|}(\cos\xi+i\sin\xi).\qquad\square$$

Example 9.9.5. *Compute*

$$\int_0^{+\infty}\frac{\cos(\xi x)}{1+x^4}\,dx,\ \xi\in\mathbb{R}.$$

Sol. — Clearly $g(x)=\frac{\cos(\xi x)}{1+x^4}$ is continuous on $\mathbb{R}$, hence locally integrable on $[0,+\infty[$ and because $|g(x)|\leqslant\frac{1}{1+x^4}\sim_{+\infty}\frac{1}{x^4}$, g is absolutely integrable at $+\infty$. Hence g is integrable on $[0,+\infty[$. To compute the integral, notice first that being g even we have

$$\int_0^{+\infty}\frac{\cos(\xi x)}{1+x^4}\,dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(\xi x)}{1+x^4}\,dx=\frac{1}{2}\mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{i\xi x}}{1+x^4}\,dx.$$

(actually: $\int_{-\infty}^{+\infty}\frac{\sin(\xi x)}{1+x^4}\,dx=0$). Therefore, if $f(z)=\frac{1}{1+z^4}$, $f\in H(\mathbb{C}\backslash\{z^4+1=0\})$. We have $z^4+1=0$ iff $z=z_k:=e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}$, $k=0,1,2,3$. It is also clear that each of these points is a first order pole for f. Finally, notice that $\lim_{z\to\infty_{\mathbb{C}}}zf(z)=0$. Hence (9.9.2) gives immediately

$$\int_{-\infty}^{+\infty}\frac{e^{i\xi x}}{1+x^4}\,dx=i2\pi\left(\operatorname{Res}\left(e^{i\xi\sharp}f,e^{i\frac{\pi}{4}}\right)+\operatorname{Res}\left(e^{i\xi\sharp}f,e^{i\frac{3}{4}\pi}\right)\right),\ \ \forall\xi\geqslant0.$$

Applying the reduced formula $\operatorname{Res}(e^{i\xi\sharp}f,z)=\frac{e^{i\xi z}}{4z^3}$ we have

$$\operatorname{Res}(e^{i\xi\sharp}f,z_0)=\frac{e^{i\xi e^{i\frac{\pi}{4}}}}{4e^{i\frac{3}{4}\pi}}=\frac{1}{4}e^{-\frac{\xi}{\sqrt{2}}}e^{i\left(\frac{\xi}{\sqrt{2}}-\frac{3}{4}\pi\right)},\ \ \operatorname{Res}(e^{i\xi\sharp}f,z_1)=\frac{e^{i\xi e^{i\frac{3}{4}\pi}}}{4e^{i\frac{\pi}{4}}}=\frac{1}{4}e^{-\frac{\xi}{\sqrt{2}}}e^{i\left(-\frac{\xi}{\sqrt{2}}-\frac{\pi}{4}\right)},$$

so

$$\int_{-\infty}^{+\infty}\frac{e^{i\xi x}}{1+x^4}\,dx=i\frac{\pi}{2}e^{-\frac{\xi}{\sqrt{2}}}\left(e^{i\left(\frac{\xi}{\sqrt{2}}-\frac{3}{4}\pi\right)}+e^{i\left(-\frac{\xi}{\sqrt{2}}-\frac{\pi}{4}\right)}\right).$$

Taking the real part we finally deduce

$$\int_0^{+\infty}\frac{\cos(\xi x)}{1+x^4}\,dx=\frac{\sqrt{2}\pi}{4}e^{-\frac{\xi}{\sqrt{2}}}\left(\cos\frac{\xi}{\sqrt{2}}+\sin\frac{\xi}{\sqrt{2}}\right),\ \forall\xi\geqslant0.$$

If $\xi<0$ we can avoid the computation and argue by symmetry. Indeed: the integral is an even function of ξ clearly. We deduce

$$\int_0^{+\infty}\frac{\cos(\xi x)}{1+x^4}\,dx=\frac{\sqrt{2}\pi}{4}e^{-\frac{|\xi|}{\sqrt{2}}}\left(\cos\frac{|\xi|}{\sqrt{2}}+\sin\frac{|\xi|}{\sqrt{2}}\right),\ \forall\xi\in\mathbb{R}.\qquad\square$$

9.9.3. Integrals involving exponentials. In this subsection, we apply the method to the problem of computing a generalized integral of a function based on e^z.

Example 9.9.6. *Discuss convergence and compute*

$$\int_0^{+\infty}\frac{x^\alpha}{1+x^2}\,dx.$$

where $\alpha\in\mathbb{R}$.

Sol. — Let $f(x) = \frac{x^\alpha}{1+x^2}$. Clearly $f \in \mathscr{C}(]0,+\infty[)$, therefore f is integrable on every interval $[a,b] \subset]0,+\infty[$. Moreover

$$f(x) \sim_{0+} x^\alpha, \quad f(x) \sim_{+\infty} \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}},$$

hence, by asymptotic comparison, f is integrable at $0+$ iff $\alpha > -1$, while it is integrable at $+\infty$ iff $2-\alpha > 1$, that is iff $\alpha < 1$. Therefore, the proposed integral converges iff $-1 < \alpha < 1$.

To compute the integral we change variable posing $x = e^t$. We have

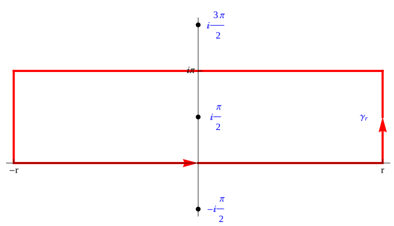
$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{1+e^{2t}} e^t\,dt = \int_{-\infty}^{+\infty} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = \lim_{r\to+\infty} \int_{-r}^{r} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = \lim_{r\to+\infty} \int_{[-r,r]} g,$$

where of course $g(z) := \frac{e^{(\alpha+1)z}}{1+e^{2z}}$, $z \in \mathbb{C}$. Singular points of g are w such that $1 + e^{2w} = 0$, that is

$$e^{2w} = -1, \iff 2w = \log 1 + i(\arg(-1) + k2\pi) = i\pi + i2\pi k,\ k \in \mathbb{Z}, \iff w = i\frac{\pi}{2} + i\pi k,\ k \in \mathbb{Z}.$$

To complete $[-r,r]$ obtaining a circuit, we consider the rectangle

$$\gamma_r := [-r,r] + [r, r+i\pi] - [-r+i\pi, r+i\pi] - [-r, -r+i\pi].$$



By the residue theorem

$$\oint_{\gamma_r} g = i2\pi \mathrm{Res}\left(g, i\frac{\pi}{2}\right).$$

Let us compute the residue. Being $g = \frac{N}{D}$, $N \neq 0$ e $D'(z) = 2e^{2z} \neq 0$ it follows that $i\frac{\pi}{2}$ is a first order pole and the residue can be computed by the reduced formula

$$\mathrm{Res}\left(g, i\frac{\pi}{2}\right) = \frac{N\left(i\frac{\pi}{2}\right)}{D'\left(i\frac{\pi}{2}\right)} = \frac{e^{i(\alpha+1)\frac{\pi}{2}}}{2e^{i\pi}} = -\frac{1}{2}e^{i(\alpha+1)\frac{\pi}{2}}.$$

On the other side,

$$\oint_{\gamma_r} = \int_{[-r,r]} + \int_{[r,r+i\pi]} - \int_{[-r+i\pi, r+i\pi]} - \int_{[-r,-r+i\pi]}.$$

Notice that

$$\int_{[-r+i\pi, r+i\pi]} g = \int_{-r}^{r} \frac{e^{(\alpha+1)(t+i\pi)}}{1+e^{2(t+i\pi)}}\,dt = e^{i(\alpha+1)\pi} \int_{-r}^{r} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = e^{i(\alpha+1)\pi} \int_{[-r,r]} g.$$

Now: we claim that the two "vertical" integrals vanish for $r \longrightarrow +\infty$. To show this, notice that, according to the triangular inequality,

$$\left|\int_{[r,r+i\pi]} g\right| = \left|\int_0^\pi \frac{e^{(\alpha+1)(r+it)}}{1+e^{2(r+it)}}\,dt\right| \leqslant \int_0^\pi \left|\frac{e^{(\alpha+1)(r+it)}}{1+e^{2(r+it)}}\right|\,dt = \int_0^\pi \frac{e^{(\alpha+1)r}}{|1+e^{2r}e^{i2t}|}\,dt.$$

Now, since $r > 0$ we have that $|e^{2r}e^{i2t}| = e^{2r} > 1$ therefore, by the triangular inequality $|z+w| \geqslant |z| - |w|$, we get $|1 + e^{2r}e^{i2t}|\ e^{2r} - 1 > 0$. Inserting this into the previous estimate we obtain

$$\left|\int_{[r,r+i\pi]} g\right| \leqslant \int_0^\pi \frac{e^{(\alpha+1)r}}{e^{2r}-1}\,dt = \pi\frac{e^{(\alpha+1)r}}{e^{2r}-1} \longrightarrow 0,\ \ r \longrightarrow +\infty,$$

being $\alpha + 1 < 2$ (recall that $-1 < \alpha < 1$). Similarly,

$$\left|\int_{[-r,-r+i\pi]} g\right| \leqslant \int_0^\pi \left|\frac{e^{(\alpha+1)(-r+it)}}{1+e^{2(-r+it)}}\right|\,dt = \int_0^\pi \frac{e^{-(\alpha+1)r}}{|1+e^{-2r}e^{i2t}|}\,dt.$$

Now: if $r > 0$ we have $|e^{-2r}e^{i2t}| = e^{-2r} < 1$ therefore, always by triangular inequality $|z+w| \geqslant |z| - |w|$, we get $|1 + e^{-2r}e^{i2t}| \geqslant 1 - e^{2r} > 0$, hence

$$\left|\int_{[-r,-r+i\pi]} g\right| \leqslant \int_0^\pi \frac{e^{-(\alpha+1)r}}{1-e^{-2r}}\,dt = \pi\frac{e^{-(\alpha+1)r}}{1-e^{-2r}} \longrightarrow 0,\ \ r \longrightarrow +\infty,$$

being $\alpha + 1 > 0$. Finally,

$$-i\pi e^{i(\alpha+1)\frac{\pi}{2}} = \lim_{r\to+\infty}\left(\int_{-r}^r \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt - e^{i(\alpha+1)\pi}\int_{-r}^r \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt\right) = \left(1 - e^{i(\alpha+1)\pi}\right)\int_{-\infty}^{+\infty} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt,$$

from which

$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\,dx = \frac{1}{2}\frac{e^{i(\alpha+1)\frac{\pi}{2}}}{e^{i(\alpha+1)\pi}-1}\frac{e^{-i(\alpha+1)\frac{\pi}{2}}}{e^{-i(\alpha+1)\frac{\pi}{2}}} = \frac{\pi}{2}\frac{1}{\sin\left((\alpha+1)\frac{\pi}{2}\right)}. \quad \square$$

Example 9.9.7. *Discuss convergence and compute the*

$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\,dx.$$

Sol. — Clearly the integrand $f \in \mathscr{C}(]0, infty[)$, thus f is integrable on every $[a,b] \subset]0,+\infty[$. For the integrability we check the behaviour of f at $x = 0, +\infty$. We have, $f(x) \sim_{0+} (\log x)^2$ which is easy to check that it is integrable. For $x \longrightarrow +\infty$, noticed that $\log x = o(x)$ we have that $f(x) = o\left(\frac{x}{1+x^4}\right) = o\left(\frac{1}{x^3}\right)$, clearly integrable. We conclude that f is integrable in generalized sense on $[0, +\infty[$.

To compute the integral, we start with a change of variable $t = \log x$. We get

$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\,dx = \int_{-\infty}^{+\infty} \frac{t^2}{1+e^{4t}}e^t\,dt = \lim_{r\to+\infty}\int_{[-r,r]} g,$$

where $g(z) := \frac{z^2e^z}{1+e^{4z}}$. Clearly $g \in H(\mathbb{C}\backslash\{e^{4z} = -1\})$. Now

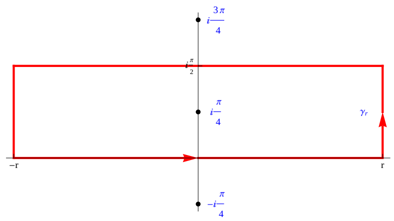
$$e^{4z} = -1, \iff 4z = i\pi + i2\pi k,\ k \in \mathbb{Z}, \iff z = i\frac{\pi}{4} + ik\frac{\pi}{2},\ k \in \mathbb{Z}.$$

Now e^{4z} does not change if we replace z with $z + i\frac{\pi}{2}$. It is therefore natural to consider the path

$$\gamma_r = [-r, r] + \left[r, r + i\frac{\pi}{2}\right] - \left[-r + i\frac{\pi}{2}, r + i\frac{\pi}{2}\right] - \left[-r, -r + i\frac{\pi}{2}\right].$$

By the residue theorem

$$\oint_{\gamma_r} g = i2\pi\mathrm{Res}\left(g, i\frac{\pi}{4}\right).$$



Clearly $z = i\frac{\pi}{4}$ is a pole of first order ($g = \frac{N}{D}$ with $D'(z) = 4e^{4z} \neq 0$ for any $z \in \mathbb{C}$ and $N(i\frac{\pi}{4}) \neq 0$), therefore,

$$\operatorname{Res}\left(g, i\frac{\pi}{4}\right) = \frac{N\left(i\frac{\pi}{4}\right)}{D'\left(i\frac{\pi}{4}\right)} = \frac{-\frac{\pi^2}{16}e^{i\frac{\pi}{4}}}{4(-1)} = \frac{\pi^2}{64}e^{i\frac{\pi}{4}}.$$

Now,

$$\int_{[-r+i\pi, r+i\pi]} g = \int_{-r}^{r} \left(t + i\frac{\pi}{2}\right)^2 e^{i\frac{\pi}{2}} \frac{e^t}{1+e^{2t}}\, dt = i\left(\oint_{-r}^{r} \frac{t^2 e^t}{1+e^{2t}}\, dt + i\pi \int_{-r}^{r} \frac{te^t}{1+e^{2t}}\, dt - \frac{\pi^2}{4}\int_{-r}^{r} \frac{e^t}{1+e^{2t}}\, dt\right).$$

Notice that the second and third integrals are easily computed. The second is the integral of an odd function over a symmetric interval (so it vanishes), because

$$\frac{-te^{-t}}{1+e^{-2t}} = -\frac{te^{-t}}{\frac{e^{2t}+1}{e^{2t}}} = -\frac{te^t}{e^{2t}+1}.$$

About the third integral, letting $r \longrightarrow +\infty$, it converges to

$$\int_{-\infty}^{+\infty} \frac{e^t}{1+e^{2t}}\, dt \overset{x=e^t}{=} \int_0^{+\infty} \frac{x}{1+x^2}\frac{1}{x}\, dx = \int_0^{+\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{4}.$$

Therefore

$$\int_{[-r,r]} - \int_{[-r+i\pi, r+i\pi]} = (1-i)\int_{-r}^{r} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^2}{4}\int_{-r}^{r} \frac{e^t}{1+e^{2t}}\, dt \longrightarrow (1-i)\int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^3}{16}.$$

Now, we claim that the two vertical integrals vanish for $r \longrightarrow +\infty$. Indeed:

$$\left|\int_{[r, r+i\frac{\pi}{2}]} g\right| = \left|\int_0^{\pi/2} g(r+it)i\, dt\right| \leqslant \int_0^{\pi/2} \frac{|r+it|^2 e^r}{|1+e^{2r}e^{i2t}|}\, dt \leqslant \frac{\pi}{2}\frac{(r+\pi)^2 e^r}{e^{2r}-1} \longrightarrow 0, \quad r \longrightarrow +\infty.$$

Similarly,

$$\left|\int_{[-r, -r+i\frac{\pi}{2}]} g\right| = \left|\int_0^{\pi/2} g(-r+it)i\, dt\right| \leqslant \int_0^{\pi/2} \frac{|-r+it|^2 e^{-r}}{|1+e^{-2r}e^{i2t}|}\, dt \leqslant \frac{\pi}{2}\frac{(r+\pi)^2 e^{-r}}{1-e^{-2r}} \longrightarrow 0, \quad r \longrightarrow +\infty.$$

Therefore

$$\frac{\pi^2}{64}e^{i\frac{\pi}{4}} = \lim_{r\to+\infty} \oint_{\gamma_r} g = (1-i)\int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^3}{16}.$$

Noticed that $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, and taking the real parts in both members we finally obtain,

$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\, dx = \int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt = \frac{\pi^2}{64}\frac{\sqrt{2}}{2}. \quad \square$$

9.10. Exercises

Exercise 9.10.1. *For each of the following power series, determine their radius of convergence:*

1. $\sum_{n=1}^{\infty} \frac{z^n}{n}$. 2. $\sum_{n=1}^{\infty} nz^n$. 3. $\sum_{n=0}^{\infty} 2^n z^n$. 4. $\sum_{n=0}^{\infty} \frac{10^n z^n}{n!}$. 5. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$.

6. $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. 7. $\sum_{n=0}^{\infty} n! z^n$. 8.

Exercise 9.10.2. *Solve the following equations in the unknown $z \in \mathbb{C}$:*

1. $\cosh^2 z + 1 = 0$. 2. $\cosh(2z) + 1 = 0$. 3. $e^{iz} = 1$. 4. $\sinh(iz + 1) = 0$.

5. $\cos z = i$. 6. $e^{z^2} = 1$. 7. $e^{iz^2} = i$. 8.

Exercise 9.10.3. *Let* log *be the principal logarithm. Show, with an example, that* $\log(zw)$ *can be different from* $\log z + \log w$.

Exercise 9.10.4. *Check, with the definition, that* $\operatorname{Im} z$, $|z|$ *and* $\bar{z}$ *are not differentiable at every point* $z \in \mathbb{C}$. *Repeat the same check by using the CR conditions.*

Exercise 9.10.5. *For each of the following $u = u(x, y)$ determine $v = v(x, y)$ in such a way $f = u + iv$ be holomorphic on $\mathbb{C}$, determining also f.*

i) $u(x, y) = x$.
ii) $u(x, y) = y$.
iii) $u(x, y) = x^2$.
iv) $u(x, y) = x^2 - y^2$.
v) $u(x, y) = x^2 + y^2$.

Exercise 9.10.6. *Let $f = f(z) \in H(\mathbb{C})$. Define*

$$g(z) := \overline{f(\bar{z})}.$$

Use CR conditions to check that $g \in H(\mathbb{C})$. Can you use the chain rule in this example? Justify your answer.

Exercise 9.10.7. *For each of the following functions, classify their singularities:*

i) $f(z) := \frac{z^2+2z+5}{(z+2)(z^2+2z+1)}$.
ii) $f(z) = \exp\left(\frac{z}{z+1}\right)$.
iii) $f(z) = \frac{e^z}{z+1}$.
iv) $f(z) = z^2 e^{\frac{1}{z-1}}$.
v) $f(z) = \frac{(z-1)^2(z+3)}{1-\sin\left(\frac{\pi z}{2}\right)}$.

Exercise 9.10.8. *Classify the singularity of $f(z) = (z - 2)\sin(\frac{1}{z+2})$ at $z = 2$; i.e., determine whether it is removable, a pole, or an essential singularity.*

EXERCISE 9.10.9. *Compute the following integrals:*

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2}\,dx$. 2. $\displaystyle\int_{0}^{+\infty} \frac{1+x^2}{1+x^4}\,dx$. 3. $\displaystyle\int_{0}^{+\infty} \frac{x^2}{(x^2+1)(x^4+1)}\,dx$.

4. $\displaystyle\int_{0}^{+\infty} \frac{\cos x}{x^2+4}\,dx$. 5. 6.

7. $\displaystyle\int_{0}^{+\infty} \frac{x^{1/3}}{x^2+9x+8}\,dx$. 8. $\displaystyle\int_{0}^{+\infty} \frac{\log x}{1+x^2}\,dx$. 9.

EXERCISE 9.10.10. *Compute the following Fourier integrals:*

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}e^{i\xi x}\,dx$. 2. $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^4}e^{i\xi x}\,dx$. 3. $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\cosh x}e^{i\xi x}\,dx$.

EXERCISE 9.10.11. *Use contour integration to verify that for $b > 0$,*

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+b^2}\,dx = \pi\frac{e^{-b}}{b},\ \forall b > 0.$$

EXERCISE 9.10.12. *Determine for which values of $a \in \mathbb{R}$ the integral*

$$I(a) := \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx$$

exists, hence compute it by using complex integration.

EXERCISE 9.10.13. *Determine for which values of $\alpha \in \mathbb{R}$ the integral*

$$I(\alpha) := \int_{0}^{+\infty} \frac{1}{(1+x)x^\alpha}\,dx,$$

converges, hence compute it by using comples integration.