Esercizi su limiti di successioni e di funzioni

1) $\lim\limits_{n\to+\infty} (-1)^n \dfrac{\cos^2 n}{n} = 0$ perché $\frac{1}{n} \to 0$ e $(-1)^n \cos^2 n$ è limitata

2) $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n-1} - 2}{(-1)^n - 2} \nexists$ perché:

$$\frac{(-1)^{n-1} - 2}{(-1)^n - 2} = \begin{cases} 3 & n \text{ pari} \\ \frac{1}{3} & n \text{ dispari} \end{cases}$$

3) $\lim\limits_{n\to+\infty} \dfrac{n^n}{e^{n^2}} = \lim\limits_{n\to+\infty} \dfrac{n^n}{e^{n\cdot n}} = \lim\limits_{n\to+\infty} \left(\dfrac{n}{e^n}\right)^n =$

$$= \lim_{n\to+\infty} e^{\overbrace{n \underbrace{\log\big(\underbrace{\tfrac{n}{e^n}}_{\to 0}\big)}_{\to -\infty}}^{\to -\infty}} = 0.$$

4) $\lim\limits_{n} \dfrac{1 + n^3 - n \operatorname{sen} n + n^2 \operatorname{sen}\left(\frac{1}{n}\right)}{\log^4 n + \sqrt{n^2 + 1}} =$

$$= \lim_{n} \frac{n^3 \left(\underbrace{\tfrac{1}{n^3}}_{\to 0} + 1 - \underbrace{\tfrac{1}{n^2} \operatorname{sen} n}_{\to 0} + \underbrace{\tfrac{1}{n} \operatorname{sen}\left(\tfrac{1}{n}\right)}_{\to 0} \right)}{n \left(\underbrace{\tfrac{\log^4 n}{n}}_{\to 0} + \sqrt{1 + \underbrace{\tfrac{1}{n^2}}_{\to 0}} \right)} = +\infty$$

infinitesimo per limitata

(scala infiniti)

5) $\lim \dfrac{4^n + a^n}{n^2 2^n + 5^n}$ $\quad a > 0$

$a > 4$

$$\frac{a^n \left(\underbrace{\left(\tfrac{4}{a}\right)^n}_{\to 0} + 1 \right)}{5^n \left(n^2 \left(\tfrac{2}{5}\right)^n + 1 \right)} \xrightarrow{n\to+\infty} \begin{cases} +\infty & a > 5 \\ 1 & a = 5 \\ 0 & 4 < a < 5 \end{cases}$$

$= \dfrac{n^2}{\left(\frac{5}{2}\right)^n} \to 0$ per scala infiniti

$a = 4$

$$\frac{2 \cdot 4^n}{5^n \left(n^2 \left(\tfrac{2}{5}\right)^n + 1 \right)} = 2 \underbrace{\left(\frac{4}{5}\right)^n}_{\to 0} \frac{1}{\frac{n^2}{\left(\frac{5}{2}\right)^n} + 1} \to 0$$

$$5^n\left(n^2\left(\tfrac{2}{5}\right)^n+1\right) \qquad \left(\frac{n^2}{\left(\frac{5}{2}\right)^n}+1\right), \quad \frac{n^2}{\left(\frac{5}{2}\right)^n}\to 0$$

$a<4$
$$\frac{4^n\left(1+\left(\frac{a}{4}\right)^n\right)}{5^n\left(\frac{n^2}{\left(\frac{5}{2}\right)^n}+1\right)} \to 0$$

Quindi
$$\lim_{n\to+\infty}\left(\;\cdots\;\right)=\begin{cases}+\infty & a>5\\ 1 & a=5\\ 0 & 0<a<5\end{cases}$$

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6) $$\lim_{n\to+\infty}\frac{n^2\log\left(1+\frac{1}{n}\right)+e^n+e^{n\log n}}{n^5+n^n}$$

$$n^2\log\left(1+\tfrac{1}{n}\right)=n\underbrace{\log\left(1+\tfrac{1}{n}\right)^n}_{\to 1}\sim n\to+\infty$$

Al numeratore il pezzo che comanda dovrebbe essere $e^{n\log n}$. Vediamo

$$e^{n\log n}\left(\frac{n^2\log\left(1+\frac{1}{n}\right)}{e^{n\log n}}+\frac{e^n}{e^{n\log n}}+1\right)$$

$$=\frac{n\log\left(1+\frac{1}{n}\right)^n}{e^{\log(n^n)}}=\frac{n\log\left(1+\frac{1}{n}\right)^n}{n^n}=$$

$$=\frac{\overbrace{\log\left(1+\frac{1}{n}\right)^n}^{\to 1}}{n^{n-1}}\longrightarrow 0$$

$$\frac{e^n}{e^{n\log n}}=\frac{1}{e^{n\log n-n}}=\frac{1}{e^{n\log n\left(1-\frac{1}{\log n}\right)}}\to 0, \quad \frac{1}{\log n}\to 0$$

Quindi

$$\lim\left(\;\cdots\;\right)=\lim e^{n\log n}\left(1+\overset{\to 0}{\cdots}\right)$$

Quindi

$$\lim_{n\to+\infty} \frac{(\qquad)}{(\qquad)} = \lim_{n\to+\infty} \frac{e^{n\log n}\left(1+\dots^{\to 0}\right)}{n^n\left(\frac{n^5}{n^n}+1\right)} =$$

$$= \lim_{n\to+\infty} \frac{\overbrace{e^{\log(n^n)}}^{n^n}}{n^n}\,\frac{\left(1+\dots^{\to 0}\right)}{\left(1+\dots_{\to 0}\right)} = 1$$

7) $$\lim_{n\to+\infty} \frac{n^{\sqrt n}}{2^n} = \lim_n \frac{e^{\sqrt n\log n}}{e^{n\log 2}} = \lim_n e^{\sqrt n\log n - n\log 2}$$

$$= \lim_{n\to+\infty} e^{\underbrace{n\log 2\left(\frac{1}{\sqrt n}\frac{\log n}{\log 2}-1\right)}_{\to -\infty}} = 0$$

$\frac{1}{\sqrt n}\frac{\log n}{\log 2} \to 0$ (scala infiniti)

8) $$\lim_n \frac{n^a - \cos n}{3n^2 + \operatorname{sen}(\sqrt n)} \qquad a=1 \quad e \quad a=3$$

$a=1$ $$\frac{n\left(1-\frac{\cos n}{n}\right)}{3n^2\left(1+\frac{\operatorname{sen}(\sqrt n)}{3n^2}\right)} \longrightarrow 0$$

$\frac{\cos n}{n}\to 0$, $\frac{\operatorname{sen}(\sqrt n)}{3n^2}\to 0$

$a=3$ $$\frac{n^3\left(1-\frac{\cos n}{n^3}\right)}{3n^2\left(1+\frac{\operatorname{sen}(\sqrt n)}{n^2}\right)} \longrightarrow +\infty$$

$\frac{\cos n}{n^3}\to 0$, $\frac{\operatorname{sen}(\sqrt n)}{n^2}\to 0$

9) $$\lim_{n\to+\infty}\left(1+\frac{1}{\sqrt n}\right)^{n^{1/3}} = \lim_n e^{n^{1/3}\log\left(1+\frac{1}{\sqrt n}\right)}$$

esponente $$\sqrt[3]{n}\,\frac{\log\left(1+\frac{1}{\sqrt n}\right)}{\frac{1}{\sqrt n}}\cdot\frac{1}{\sqrt n} = n^{\frac13-\frac12}\,\frac{\log\left(1+\frac{1}{\sqrt n}\right)}{\frac{1}{\sqrt n}} =$$

$\frac{\log\left(1+\frac{1}{\sqrt n}\right)}{\frac{1}{\sqrt n}} \to 1$

$\left(\frac{\log(1+x)}{x} \xrightarrow[x\to 0]{} 1\right)$

$$= \frac{1}{n^{1/6}}\,\frac{\log\left(1+\frac{1}{\sqrt n}\right)}{\frac{1}{\sqrt n}} \longrightarrow 0$$

$\frac{\log\left(1+\frac{1}{\sqrt n}\right)}{\frac{1}{\sqrt n}} \to 1$

Quindi

$$\lim_{n\to+\infty}(\qquad) = e^{(\ )} = 1$$

$$\lim_{n\to+\infty} (\quad) = e^{\ '} = 1$$

10) $$\lim_{n\to+\infty} \left(\frac{n^2+3n-2}{n^2-4n}\right)^{n^2-n+3}$$

base $\frac{n^2+3n-2}{n^2-4n} \xrightarrow[n\to+\infty]{} 1$ quindi cerco di scriverlo come $1+\underbrace{(\dots)}_{\to 0}$

$$\frac{n^2+3n-2}{n^2-4n} = \frac{n^2-4n+4n+3n-2}{n^2-4n} =$$

$$= 1 + \frac{7n-2}{n^2-4n}$$

$$\left(\frac{n^2+3n-2}{n^2-4n}\right)^{n^2-n+3} = e^{\underbrace{(n^2-n+3)\log\left(1+\frac{7n-2}{n^2-4n}\right)}_{\text{esponente}}}$$

$\frac{7n-2}{n^2-4n} \to 0$

$\frac{\log(1+x)}{x} \xrightarrow[x\to 0]{} 1$

esponente $= (n^2-n+3)\dfrac{\log\left(1+\frac{7n-2}{n^2-4n}\right)}{\frac{7n-2}{n^2-4n}} \cdot \dfrac{7n-2}{n^2-4n}$

$\to 1$

e

$$(n^2-n+3)\frac{(7n-2)}{n^2-4n} \to +\infty$$

Quindi $e^{(\text{esponente})} \xrightarrow[n\to+\infty]{} +\infty$.

11) $$\lim_{x\to 0} \frac{e^{2x}-e^{-2x}}{tg(3x)} = \lim_{x\to 0} \frac{e^{-2x}(e^{4x}-1)}{tg(3x)} =$$

limiti notevoli $\frac{e^y-1}{y} \to 1$ se $y\to 0$

$\frac{tg\, y}{y} \to 1$ se $y \to 0$

$$= \lim_{x\to 0} e^{-2x} \frac{(e^{4x}-1)}{4x} \cdot 4x \cdot \frac{3x}{tg(3x)} \cdot \frac{1}{3x} =$$

$\to 1 \ (x\to 0)$ $\quad\to 1\ (x \to 0)$

$$= \lim_{x\to 0} e^{-2x} \frac{4}{3} \left(\frac{e^{4x}-1}{4x}\right) \cdot \left(\frac{3x}{tg(3x)}\right) = \frac{4}{3}$$

$$= \lim_{x\to 0} \underbrace{e^{\ldots}}_{\to 1} \cdot \frac{1}{3} \underbrace{\left(\frac{e^{\ldots}-1}{4x}\right)}_{\to 1} \cdot \underbrace{\left(\frac{\ldots}{tg(3x)}\right)}_{\to 1} \ldots \frac{\ldots}{3}$$

12) $\displaystyle \lim_{x\to 0} \frac{\lg(1+\operatorname{sen} x)}{x} =$

limite notevole $\displaystyle \lim_{y\to 0} \frac{\lg(1+y)}{y} = 1$

ponendo $y = \operatorname{sen} x$

$$= \lim \underbrace{\frac{\lg(1+\operatorname{sen} x)}{\operatorname{sen} x}}_{\to 1} \cdot \underbrace{\frac{\operatorname{sen} x}{x}}_{\to 1} = 1$$

infatti

$$\lim_{x\to 0} \frac{\lg(1+\operatorname{sen} x)}{\operatorname{sen} x} \underset{y=\operatorname{sen} x}{=} \lim_{y\to 0} \frac{\log(1+y)}{y} = 1$$

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13) $\displaystyle \lim_{x\to 0} \frac{e^{1-\cos x}-1}{\log(1+3x)} =$

$x\to 0 \quad 1-\cos x = y \to 0$

uso

$\dfrac{e^{y}-1}{y} \to 1$ se $y\to 0$

e

$\dfrac{\lg(1+y)}{y}$

$$= \lim_{x\to 0} \frac{e^{1-\cos x}-1}{1-\cos x} \cdot \frac{\ldots}{\log(1+3x)} \cdot \frac{3x}{3x}$$

$$\lim_{x\to 0} \frac{e^{1-\cos x}-1}{1-\cos x} = \lim_{y\to 0} \frac{e^{y}-1}{y} = 1$$

$y = 1-\cos x \to 0$
se $x\to 0$

$$\lim_{x\to 0} \frac{\log(1+3x)}{3x} = \lim_{y\to 0} \frac{\log(1+y)}{y} = 1$$

$y = 3x \to 0$
se $x\to 0$

Quindi $\displaystyle \lim_{x\to 0} \frac{e^{1-\cos x}-1}{\log(1+3x)} = \lim_{x\to 0} \underbrace{\frac{e^{1-\cos x}-1}{1-\cos x}}_{\to 1} \cdot \frac{1-\cos x}{3x} \cdot \underbrace{\frac{3x}{\lg(1+3x)}}_{\to 1}$

Inoltre $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{3x} = \lim_{x\to 0} \underbrace{\frac{1-\cos x}{x^2}}_{\to 1/2} \cdot \frac{x^2}{3x} = 0$

se $x\to 0$

Quindi $\displaystyle \lim_{x\to 0} (\ldots) = 0 \cdot 1 \cdot 1 = 0$

$x \to 0$

14) $$\lim_{x\to+\infty}\left(\frac{x^2+3x-1}{x^2+1}\right)^x$$

noto che $\displaystyle\lim_{x\to+\infty}\frac{x^2+3x-1}{x^2+1}=1$ quindi cerco di scrivere

$$\frac{x^2+3x-1}{x^2+1}=1+\underbrace{(\dots)}_{\to 0 \quad x\to+\infty}$$

$$\frac{x^2+3x-1}{x^2+1}=\frac{x^2+1-1+3x-1}{x^2+1}=1+\underbrace{\frac{3x-2}{x^2+1}}_{\to 0 \quad x\to+\infty}$$

Quindi

$$\lim_{x\to+\infty}\left(\frac{x^2+3x-1}{x^2+1}\right)^x=\lim_{x\to+\infty}e^{x\log\left(1+\frac{3x-2}{x^2+1}\right)}$$

facciamo il limite dell'esponente

$$\lim_{x\to+\infty}x\log\left(1+\frac{3x-2}{x^2+1}\right)=$$

limite notevole $\displaystyle\lim_{y\to0}\frac{\log(1+y)}{y}=1$

$$=\lim_{x\to+\infty}x\,\underbrace{\frac{\log\left(1+\frac{3x-2}{x^2+1}\right)}{\frac{3x-2}{x^2+1}}}_{\to 1 \quad x\to+\infty}\cdot\frac{3x-2}{x^2+1}$$

(con sostituzione $y=\frac{3x-2}{x^2+1}$)

Inoltre

$$\lim_{x\to+\infty}\frac{x\cdot(3x-2)}{x^2+1}=3$$

Quindi $\displaystyle\lim_{x\to+\infty}x\log\left(1+\frac{3x-2}{x^2+1}\right)=3\cdot1=3$

Quindi $\displaystyle\lim_{x\to+\infty}e^{(\ )}=e^3.$

15) $$\lim_{x\to0}(1+\operatorname{sen}x)^{2/x}=\lim_{x\to0}e^{\frac{2}{x}\log(1+\operatorname{sen}x)}$$

Calcolo

$$\lim_{x\to0}\frac{2}{x}\log(1+\operatorname{sen}x)=\lim_{x\to0}\frac{2}{x}\cdot\underbrace{\frac{\log(1+\operatorname{sen}x)}{\operatorname{sen}x}}_{\to1}\operatorname{sen}x$$

$$= \lim_{x \to 0} 2 \, \frac{\log(1 + \operatorname{sen} x)}{\operatorname{sen} x} \cdot \frac{\operatorname{sen} x}{x} = 2$$

$$\frac{\operatorname{sen} x}{x} \to 1$$

$$\frac{\log(1+y)}{y} \xrightarrow[y \to 0]{} 1$$

$\to 1$ perché $y = \operatorname{sen} x \to 0$ se $x \to 0$