Soluzioni esercizi su integrali impropri

Es 1 Notiamo che $\frac{1}{e^{2x}+1} \sim \frac{1}{e^{2x}}$ per $x \to +\infty$

$$\int_0^{+\infty} \frac{1}{e^{2x}}\,dx = \lim_{M\to+\infty} \int_0^M e^{-2x}\,dx = \lim_{M\to+\infty} \left[-\frac{1}{2}e^{-2x}\right]_0^M = \frac{1}{2}$$

Dunque per il ~~prin~~ criterio del confronto asintotico $\int_0^{+\infty} \frac{1}{e^{2x}+1}\,dx$ converge

$$\int \frac{1}{e^{2x}+1}\,dx = \left(\begin{array}{l} y = e^{2x} \\ x = \frac{1}{2}\lg y \end{array} \quad dx = \frac{1}{2}\frac{1}{y}\,dy \right) = \int \frac{1}{(y+1)}\frac{1}{2y}\,dy$$

$$= \frac{1}{2}\int \frac{1}{y(y+1)}\,dy = \frac{1}{2}\left[\int \frac{1}{y}\,dy - \int \frac{1}{y+1}\,dy\right] =$$

$$\left[\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}\right]$$

$$= \frac{1}{2}\left(\lg|y| - \lg|y+1|\right) + c = \frac{1}{2}\lg\frac{|y|}{|y+1|} + c = \frac{1}{2}\lg\left(\frac{e^{2x}}{e^{2x}+1}\right) + c$$

$$\int_0^{+\infty} \frac{1}{1+e^{2x}}\,dx = \lim_{M\to+\infty}\left[\frac{1}{2}\lg\left(\frac{e^{2x}}{e^{2x}+1}\right)\right]_0^M = \lim_{M\to+\infty} \frac{1}{2}\lg\underbrace{\left(\frac{e^M}{e^M+1}\right)}_{1} +$$

$$-\frac{1}{2}\lg\left(\frac{1}{2}\right) = -\frac{1}{2}\lg\left(\frac{1}{2}\right) = +\frac{1}{2}\lg 2 = \lg\sqrt{2}.$$

2) Notiamo che

$$\frac{5}{x^\alpha(14 + 9\lg x + \lg^2 x)} \sim \frac{1}{x^\alpha \lg^2 x}$$

per $x \to +\infty$

Dunque l'integrale converge se $\alpha \geq 1$

Se $\alpha = 1$

$$\int \frac{5}{x(14+9\lg x+\lg^2 x)}\,dx = \begin{pmatrix} y=\lg x \\ dy=\frac{1}{x}dx \end{pmatrix} = \int \frac{5}{14+9y+y^2}\,dy$$

$$\frac{5}{14+9y+y^2} = \frac{5}{(y+2)(y+7)} = \frac{A}{y+2} + \frac{B}{y+7} \qquad \begin{array}{l} A=1 \\ B=-1 \end{array}$$

$$= \int \frac{1}{y+2}\,dy - \int \frac{1}{y+7}\,dy = \lg|y+2| - \lg|y+7| + c =$$

$$= \lg\left|\frac{y+2}{y+7}\right| + c = \lg\left|\frac{\lg x+2}{\lg x+7}\right| + c$$

$$\int_1^{+\infty} \frac{5}{x(14+9\lg x+\lg^2 x)}\,dx = \lim_{M\to+\infty}\left[\lg\left|\frac{\lg x+2}{\lg x+7}\right|\right]_1^M =$$

$$= \lim_{M\to+\infty} \underbrace{\lg\left(\frac{\lg M+2}{\lg M+7}\right)}_{\downarrow 0} - \lg\left(\frac{2}{7}\right) = -\lg\frac{2}{7} = \lg\frac{7}{2}.$$

Es 3

Notiamo che

$$\frac{\sin x}{\lg(1+\sqrt{x})\,(e^{x^\alpha}-1)} \sim \frac{x}{\sqrt{x}\;x^\alpha} = \frac{1}{x^{\alpha+\frac{1}{2}-1}} = \frac{1}{x^{\alpha-\frac{1}{2}}}$$

per $x \to 0$.

Dunque l'integrale converge se e solo se $\alpha - \frac{1}{2} < 1$

$\iff \alpha < \frac{3}{2}$.

Es 4

$$\int_3^{7/2} \frac{\sin(x-3)^\alpha\,(x-4)}{(x-3)^2\,\lg(x-2)}\,dx = \begin{pmatrix} y=x-3 \\ dy=dx \end{pmatrix} =$$

$$= \int_0^{1/2} \frac{\sin(y^\alpha)\,(-1)}{y^2 \cdot \lg(1+y)}\,dy.$$

Ora notiamo che

$$\frac{\sin y^\alpha}{y^2 \lg(1+y)} \sim \frac{y^\alpha}{y^2\, y} = \frac{1}{y^{3-\alpha}} \quad \text{per } y \to 0$$

e dunque l'integrale converge se e solo se $3-\alpha < 1$

$\Leftrightarrow \alpha > 2$.

Es 5 $\displaystyle\int_0^{+\infty} \frac{x \operatorname{arctg} x}{x^\alpha}\,dx = \int_0^1 \frac{x \operatorname{arctg} x}{x^\alpha}\,dx + \int_1^{+\infty} \frac{x \operatorname{arctg} x}{x^\alpha}\,dx$

per $x \to 0$ $\displaystyle\frac{x \operatorname{arctg} x}{x^\alpha} \sim \frac{x \cdot x}{x^\alpha} = \frac{1}{x^{\alpha-2}}$

dunque $\displaystyle\int_0^1 \frac{x \operatorname{arctg} x}{x^\alpha}\,dx$ converge $\Leftrightarrow \alpha - 2 < 1 \Leftrightarrow \alpha < 3$

per $x \to +\infty$ $\displaystyle\frac{x \operatorname{arctg} x}{x^\alpha} \sim \frac{x}{x^\alpha} = \frac{1}{x^{\alpha-1}}$

dunque $\displaystyle\int_1^{+\infty} \frac{x \operatorname{arctg} x}{x^\alpha}\,dx$ converge $\Leftrightarrow \alpha - 1 > 1 \Leftrightarrow \alpha > 3$

In conclusione l'integrale converge se $\boxed{2 < \alpha < 3}$

6) Notiamo che per $x \to +\infty$

$$x^\alpha\left(1 - \cos\frac{1}{x}\right) \sim x^\alpha \cdot \frac{1}{x^2} = \frac{1}{x^{2-\alpha}}$$

dunque l'integrale converge se e solo se $2-\alpha > 1$

$\boxed{\alpha < 1}$

Se $\alpha = -3$ calcolo

$$\int x^{-3}\left(1 - \cos\frac{1}{x}\right)dx = \left(\begin{array}{l} y = \frac{1}{x} \\ dy = -x^{-2}dx \end{array}\right) =$$

$$= -\int y(1-\cos y)\,dy = \int (y\cos y - y)\,dy =$$

$$\int y\cos y\,dy = \text{per parti} = y\sin y - \int \sin y\,dy =$$
$$= y\sin y + \cos y + c$$

Dunque

$$\int (y\cos y - y)\,dy = y\sin y + \cos y - \frac{y^2}{2} + C =$$
$$= \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x} - \frac{1}{2x^2} + C.$$

$$\int_{2/\pi}^{+\infty} x^{-3}\left(1-\cos\frac{1}{x}\right)dx = \lim_{M\to+\infty}\left[\frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x} - \frac{1}{2x^2}\right]_{2/\pi}^{M} =$$

$$= \lim_{M\to+\infty} \underbrace{\frac{1}{M}\sin\frac{1}{M}}_{\to 0} + \underbrace{\cos\frac{1}{M}}_{\to 1} - \underbrace{\frac{1}{2M^2}}_{\to 0} - \underbrace{\frac{\pi}{2}\sin\frac{\pi}{2}} - \underbrace{\cos\frac{\pi}{2}}_{0} +$$
$$+ \frac{1}{2}\frac{\pi^2}{4} = 1 - \pi/2 + \frac{\pi^2}{8}.$$