Abbiamo visto che se X e Y congiuntamente continue

e $\begin{pmatrix} Z \\ W \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ allora

$$f_{Z,W}(z,w) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\right)$$

Da ciò abbiamo dedotto che

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\,dx = \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\,dy$$

Se X e Y sono indipendenti

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\, dx = \int_{-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$= (f_X * f_Y)(z)$$ convoluzione di f_X e f_Y

Esempio 1 $X, Y \sim Exp(\lambda)$ indipendenti.

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\, dx \qquad f_X(x) = f_Y(x) = \lambda e^{-\lambda x}\, 1\!\!1_{[0,+\infty)}(x)$$

Sicuramente $f_{X+Y}(z) = 0$ se $z < 0$

$z \geq 0$

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} \mathbb{1}_{[0,+\infty)}(x) \mathbb{1}_{[0,+\infty)}(z-x)\, dx$$

$$= \lambda^2 \int_0^z e^{-\lambda z}\, dx = \lambda^2 z e^{-\lambda z}$$

In conclusione: $f_{X+Y}(z) = \lambda^2 z e^{-\lambda z} \mathbb{1}_{[0,+\infty)}(z)$

cioè $X+Y \sim \Gamma(2,\lambda)$. In altre parole

$X, Y \sim \Gamma(1,\lambda)$ indip $\Longrightarrow X+Y \sim \Gamma(2,\lambda)$

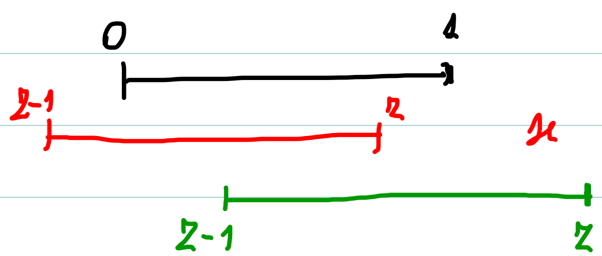
Si può dimostrare, lo faremo con un altro metodo, che se $X \sim \Gamma(\alpha,\lambda)$ e $Y \sim \Gamma(\beta,\lambda)$ indipendenti, allora $X+Y \sim \Gamma(\alpha+\beta,\lambda)$

Esempio $X, Y \sim U(0,1)$ indipendenti

$f_{X+Y}(z) = 0$ se $z<0$ o $z>2$. Per $0 \le z \le 2$

$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} 1\!\!1_{[0,1]}(x)\, 1\!\!1_{[0,1]}(z-x)\, dx$$

$$1\!\!1_{[0,1]}(x)\, 1\!\!1_{[0,1]}(z-x) = 1 \iff \begin{cases} 0 \le x \le 1 \\ z-1 \le x \le z \end{cases}$$

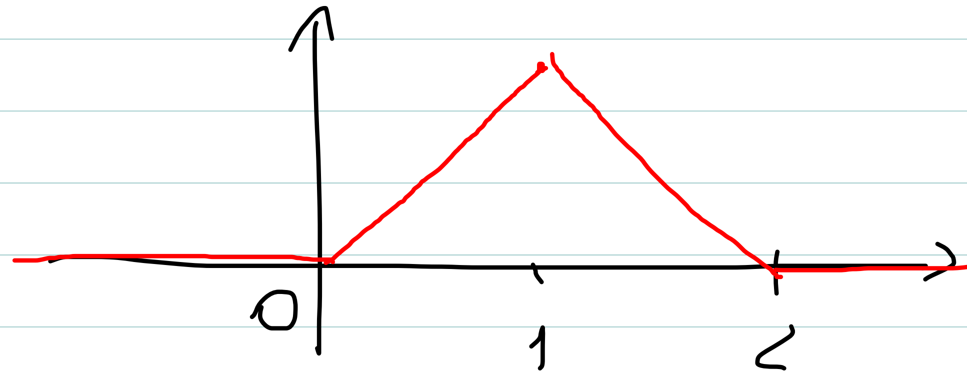


se $z \le 1$

$z > 1$

Quindi
$$f_{X+Y}(z) = \begin{cases} \int_0^z dx = z & 0 \le z \le 1 \\ \int_{z-1}^1 dx = 2 - z & 1 < z \le 2 \end{cases}$$

Grafico di f_{X+Y}:



Esercizio Siano $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ indipendenti

Calcolare $P(X<Y)$

$$\{X<Y\} = \{(X,Y) \in B\} \quad \text{dove} \quad B = \{(x,y) : x<y\}$$

$$\Rightarrow P(X<Y) = \iint_B f_{X,Y}(x,y)\,dx\,dy = \iint f_{X,Y}(x,y)\,\mathbb{1}_B(x,y)\,dx\,dy$$

$$= \iint \lambda e^{-\lambda x} \mathbb{1}_{[0,+\infty)}(x) \, \mu e^{-\mu y} \mathbb{1}_{[0,+\infty)}(y) \, \mathbb{1}_B(x,y) \, dx \, dy =$$

$$= \int_{-\infty}^{+\infty} \lambda \mu \, e^{-\lambda x} \mathbb{1}_{[0,+\infty)}(x) \left[\int_{-\infty}^{+\infty} e^{-\mu y} \mathbb{1}_{[0,+\infty)}(y) \, \mathbb{1}_B(x,y) \, dy \right] dx$$

$$= \lambda \mu \int_0^{+\infty} e^{-\lambda x} \left[\int_x^{+\infty} e^{-\mu y} \, dy \right] dx = \lambda \mu \int_0^{+\infty} e^{-\lambda x} \left[-\frac{1}{\mu} e^{-\mu y} \Big|_x^{+\infty} \right] dx$$

$$= \lambda \mu \int_0^{+\infty} e^{-\lambda x} \frac{1}{\mu} e^{-\mu x} \, dx = \lambda \int_0^{+\infty} e^{-(\lambda+\mu) x} \, dx = \frac{\lambda}{\lambda + \mu}$$

Esercizio $X, Y \sim N(0,1)$ indipendenti, $\sigma > 0$

$$Z = X + \sigma Y \qquad W = X - \frac{Y}{\sigma}$$

Mostrare che Z e W sono due normali indipendenti

Nota: $Cov(Z,W) = Cov(X + \sigma Y, X - \frac{1}{\sigma} Y)$

$= Var(X) - Var(Y) = 0 \Rightarrow Z$ e W sono scorrelate

$$\begin{pmatrix} Z \\ W \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{dove} \quad A = \begin{pmatrix} 1 & \sigma \\ 1 & -\frac{1}{\sigma} \end{pmatrix}$$

$$\det(A) = -\frac{1}{\sigma} - \sigma = -\frac{1+\sigma^2}{\sigma} \neq 0$$

$$\begin{cases} X + \sigma Y = Z \\ X - \frac{1}{\sigma} Y = W \end{cases} \qquad \left(\sigma + \frac{1}{\sigma}\right) Y = Z - W$$

$$(1+\sigma^2) X = Z + \sigma^2 W$$

$$X = \frac{1}{1+\sigma^2} Z + \frac{\sigma^2}{1+\sigma^2} W$$

$$Y = \frac{\sigma}{1+\sigma^2} Z - \frac{\sigma}{1+\sigma^2} W$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{1+\sigma^2} & \frac{\sigma^2}{1+\sigma^2} \\ \frac{\sigma}{1+\sigma^2} & -\frac{\sigma}{1+\sigma^2} \end{pmatrix}$$

cioè $A^{-1} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma^2} z + \frac{\sigma^2}{1+\sigma^2} w \\ \frac{\sigma}{1+\sigma^2} z - \frac{\sigma}{1+\sigma^2} w \end{pmatrix}$

$$f_{Z,W}(z,w) = \frac{1}{|\det(A)|} f_{X,Y}\left(A^{-1} \begin{pmatrix} z \\ w \end{pmatrix}\right) \qquad f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}}$$