$X, Y \sim N(0,1)$ indipendenti $\quad Z = X + \sigma Y$

$W = X - \frac{1}{\sigma} Y \qquad \sigma > 0$

$$\begin{pmatrix} Z \\ W \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = \begin{pmatrix} 1 & \sigma \\ 1 & -\frac{1}{\sigma} \end{pmatrix} \qquad \det A = -\frac{1}{\sigma} - \sigma = -\frac{1+\sigma^2}{\sigma}$$

$$A^{-1}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\sigma^2} z + \frac{\sigma^2}{1+\sigma^2} w \\ \frac{\sigma}{1+\sigma^2} z - \frac{\sigma}{1+\sigma^2} w \end{pmatrix} \qquad f_{Z,W}(z,w) = \frac{1}{|\det A|} f_{X,Y}\left(A^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\right)$$

$$f_{Z,W}(z,w) = \frac{\sigma}{1+\sigma^2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{1}{1+\sigma^2} z + \frac{\sigma^2}{1+\sigma^2} w\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{\sigma}{1+\sigma^2} z - \frac{\sigma}{1+\sigma^2} w\right)^2\right]$$

$$= \frac{\sigma}{1+\sigma^2}\left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left[-\frac{1}{2}\left(\frac{1}{(1+\sigma^2)^2} + \frac{\sigma^2}{(1+\sigma^2)^2}\right) z^2\right] \exp\left[-\frac{1}{2}\left(\frac{\sigma^4}{(1+\sigma^2)^2} + \frac{\sigma^2}{(1+\sigma^2)^2}\right) w^2\right]$$

$$= \frac{\sigma}{1+\sigma^2}\left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left[-\frac{1}{2}\left(\frac{1}{(1+\sigma^2)^2}+\frac{\sigma^2}{(1+\sigma^2)^2}\right)z^2\right]\exp\left[-\frac{1}{2}\underbrace{\left(\frac{\sigma^4}{(1+\sigma^2)^2}+\frac{\sigma^2}{(1+\sigma^2)^2}\right)}_{\sigma^2/(1+\sigma^2)}w^2\right]$$

$$= \frac{\sigma}{1+\sigma^2}\left(\frac{1}{\sqrt{2\pi}}\right)^2 \exp\left[-\frac{z^2}{2(1+\sigma^2)}\right]\exp\left[-\frac{w^2}{2\left(1+\frac{1}{\sigma^2}\right)}\right]$$

$$= \underbrace{\frac{1}{\sqrt{2\pi(1+\sigma^2)}}\exp\left[-\frac{z^2}{2(1+\sigma^2)}\right]}_{\text{densità di } N(0,\,1+\sigma^2)}\ \underbrace{\frac{1}{\sqrt{2\pi\left(1+\frac{1}{\sigma^2}\right)}}\exp\left[-\frac{w^2}{2\left(1+\frac{1}{\sigma^2}\right)}\right]}_{\text{densità di } N\left(0,\,1+\frac{1}{\sigma^2}\right)}$$

Quindi: Z e W sono indipendenti

$Z \sim N(0, 1+\sigma^2)$ $\quad W \sim N(0, 1+\frac{1}{\sigma^2})$

Mostriamo ora che, in generale, la somma di due normali indipendenti è Normale.

$X \sim N(\mu_X, \sigma_X^2)$ $\qquad$ $Y \sim N(\mu_Y, \sigma_Y^2)$ $\qquad$ indipendenti

$\bar{X} = \dfrac{X - \mu_X}{\sigma_X}$ $\qquad$ $\bar{Y} = \dfrac{Y - \mu_Y}{\sigma_Y}$ $\qquad$ $\bar{X}, \bar{Y} \sim N(0,1)$ indipendenti

$$X + Y = \sigma_X \bar{X} + \mu_X + \sigma_Y \bar{Y} + \mu_Y = (\mu_X + \mu_Y) + \sigma_X \underbrace{\left[\bar{X} + \frac{\sigma_Y}{\sigma_X} \bar{Y} \right]}_{\text{v.a. Normale}}$$

$\Rightarrow X + Y$ è normale

$$\Rightarrow X+Y \sim N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$$

Massimo e minimo di v.a. indipendenti

Siano $X_1, X_2, \dots, X_n$ v.a. indipendenti. Siano $F_1, \dots, F_n$ le loro funzioni di ripartizione

$$W = \min(X_1, X_2, \dots, X_n) \qquad Z = \max(X_1, X_2, \dots, X_n)$$

Determiniamo le distribuzioni di Z e W. Più precisamente, troviamo le funzioni di ripartizione di Z e W.

$$F_Z(z) = P(Z \le z) = P(\max(X_1, X_2, \dots, X_n) \le z) =$$

$$= P(X_1 \le z, X_2 \le z, \dots, X_n \le z) \quad (\text{indipendenza delle } X_i)$$

$$= P(X_1 \le z) P(X_2 \le z) \cdots P(X_n \le z) = F_1(z) F_2(z) \cdots F_n(z)$$

<u>Esempio</u> $X_1, \dots, X_n \sim U(0,1)$ indipendenti

$$F_i(x) = \begin{cases} 1 & x \ge 1 \\ x & 0 \le x \le 1 \\ 0 & x < 0 \end{cases} \implies F_Z(z) = \begin{cases} 1 & z \ge 1 \\ z^n & \\ 0 & z < 0 \end{cases}$$

$$\Rightarrow f_Z(z) = n z^{n-1} \mathbb{1}_{[0,1]}(z)$$

$$E(Z) = \int_0^1 z f_Z(z) = n \int_0^1 z^n dz = \frac{n}{n+1}$$

Torniamo al caso generale.

$$F_W(w) = P(\min(X_1, X_2, \dots, X_n) \le w) = 1 - P(\min(X_1, X_2, \dots, X_n) > w)$$

$$= 1 - P(X_1 > w, X_2 > w, \dots, X_n > w) = 1 - P(X_1 > w) P(X_2 > w) \cdots P(X_n > w)$$

$$= 1 - (1 - F_1(w))(1 - F_2(w)) \cdots (1 - F_n(w))$$

Esempio $X_1, X_2, \dots, X_n$ indipendenti $X_i \sim \mathrm{Exp}(\lambda_i)$

$W = \min(X_1, \dots, X_n)$. Riscriviamo la formula precedente

$$1 - F_W(w) = \prod_{i=1}^{n} \left(1 - F_{X_i}(w)\right) \qquad (w \geq 0)$$

$$1 - F_{X_i}(w) = P(X_i > w) = e^{-\lambda_i w}$$

$$\Rightarrow \quad 1 - F_W(w) = \prod_{i=1}^{n} e^{-\lambda_i w} = e^{-(\lambda_1 + \dots + \lambda_n) w}$$

$$\Rightarrow \quad W \sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

<u>Esempio</u> $X \sim \mathcal{G}e(p_1)$ $Y \sim \mathcal{G}e(p_2)$ $W = \min(X, Y)$

$n \geq 0$ $P(W > n) = P(X > n)\, P(Y > n) = (1-p_1)^n (1-p_2)^n$

$$P(W = n) = P(W > n-1) - P(W > n)$$

$$= (1-p_1)^{n-1}(1-p_2)^{n-1} - (1-p_1)^n (1-p_2)^n$$

$$= \underbrace{\left(1 - (1-p_1)(1-p_2)\right)}_{q} \left[(1-p_1)(1-p_2)\right]^{n-1} = q(1-q)^{n-1}$$

$$\Longrightarrow W \sim \mathcal{G}e(q)$$